On the Queue Number of Planar Graphs

GD2021

Michael A. Bekos[1] Martin Gronemann[2] Chrysanthi N. Raftopoulou[3]

1. University of Tübingen, Germany
2. Osnabrück University,Germany
3. National Technical University of Athens, Greece

Tübingen, 16/09/2021

Definitions

Linear Layout

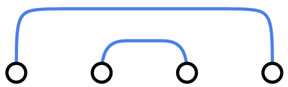
- vertices along the spine
- edges on pages

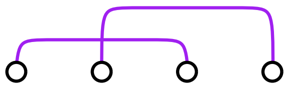
Definitions

Linear Layout

- vertices along the spine
- edges on pages


necklace


rainbow


twist

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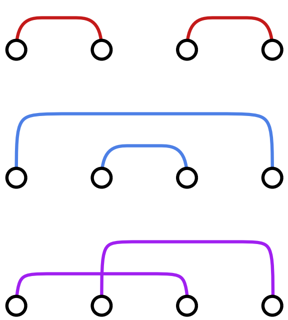

	queue layout
necklace	yes
rainbow	no
twist	yes

Definitions

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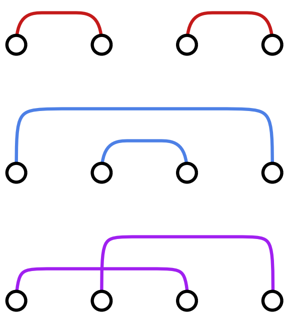

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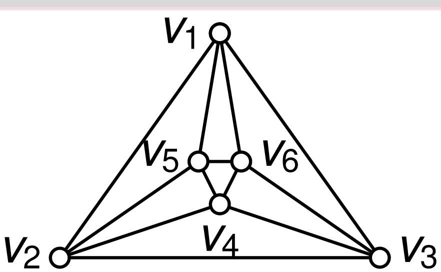
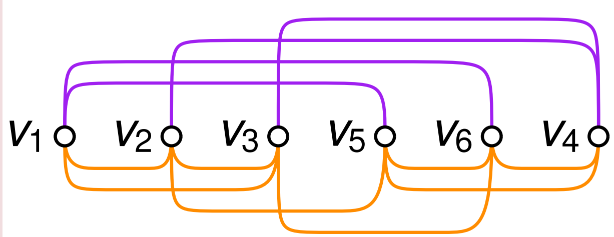


octahedron



queue layout

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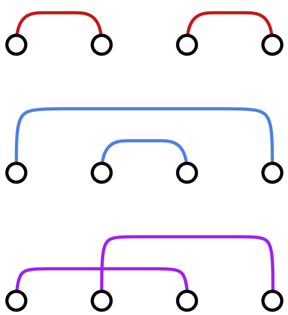

	queue layout
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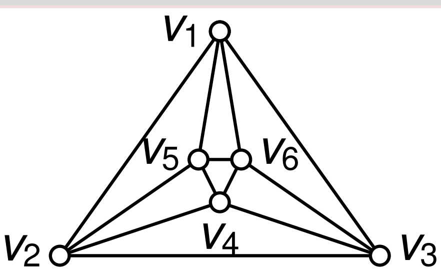
Queue number of graph

minimum number of pages
over all queue layouts

Queue number of graph family

maximum queue number
over all elements



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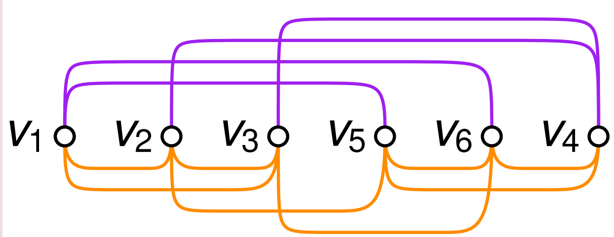

queue layout

On the Queue Number Of Planar Graphs
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Problem: Queue number of Planar Graphs

Theorem[1]:

Planar graphs have queue number at most 49.

[1] Dujmović, Joret, Micek, Morin, Ueckerdt, Wood *Planar graphs have bounded queue-number* (2020)

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Theorem[1]:

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Improvement:

Planar graphs have queue number at most 42.

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Improvement:

Planar graphs have queue number at most 42.

Idea:

- Exploit planarity
- Modify existing technique

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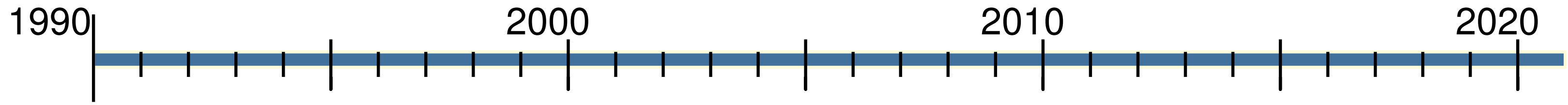
Timeline



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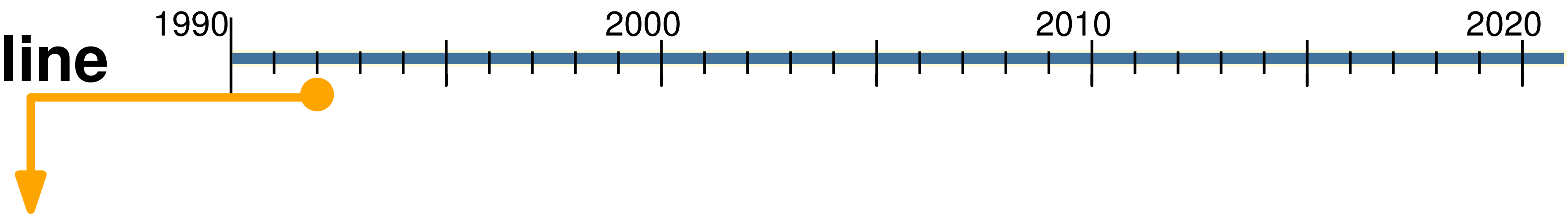

1992 [1]

$O(\sqrt{n})$

Conjecture[1]**:**

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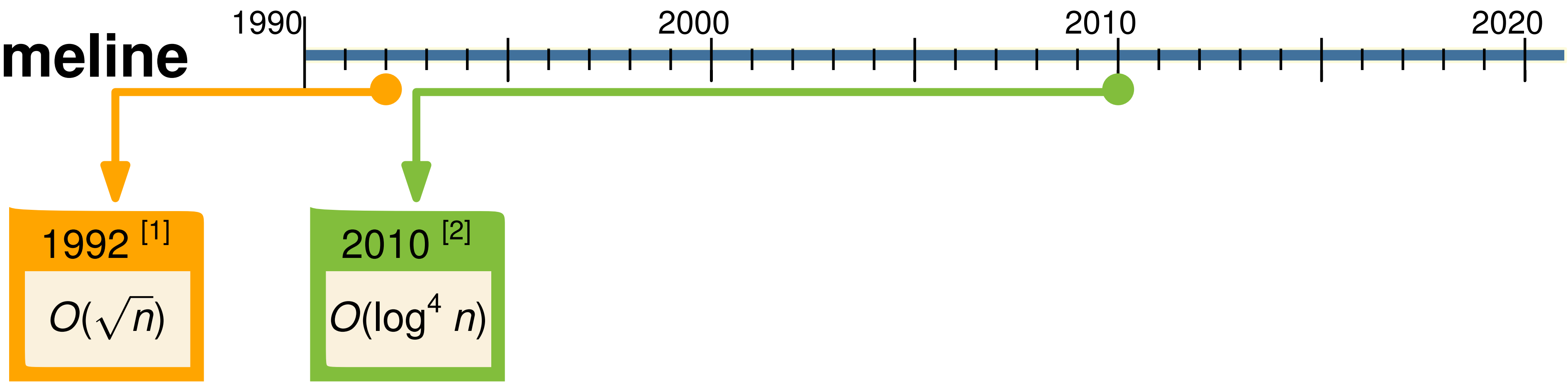

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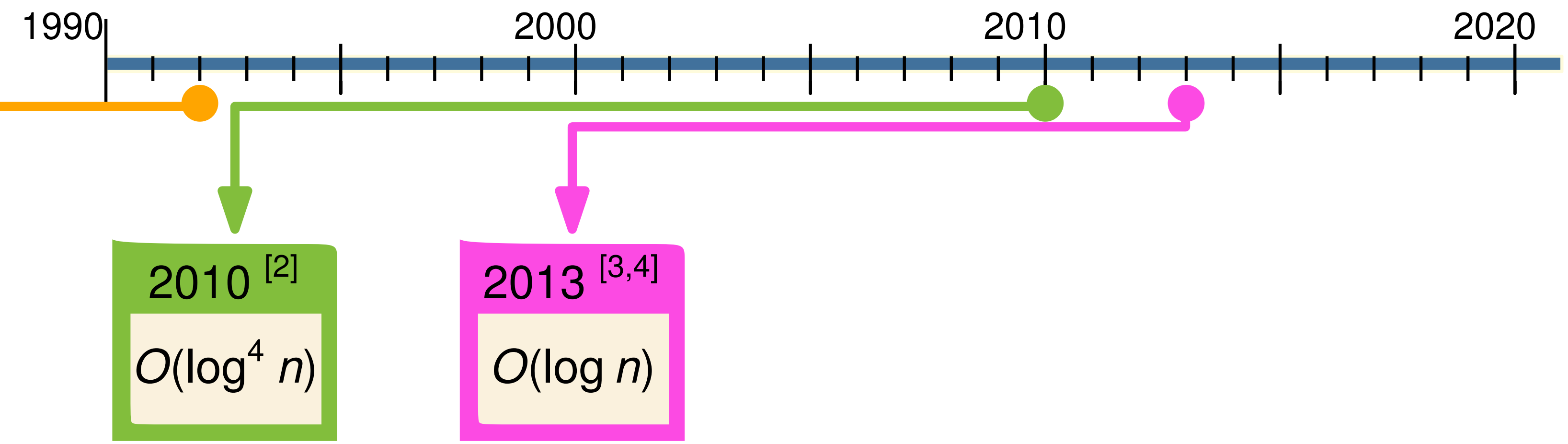

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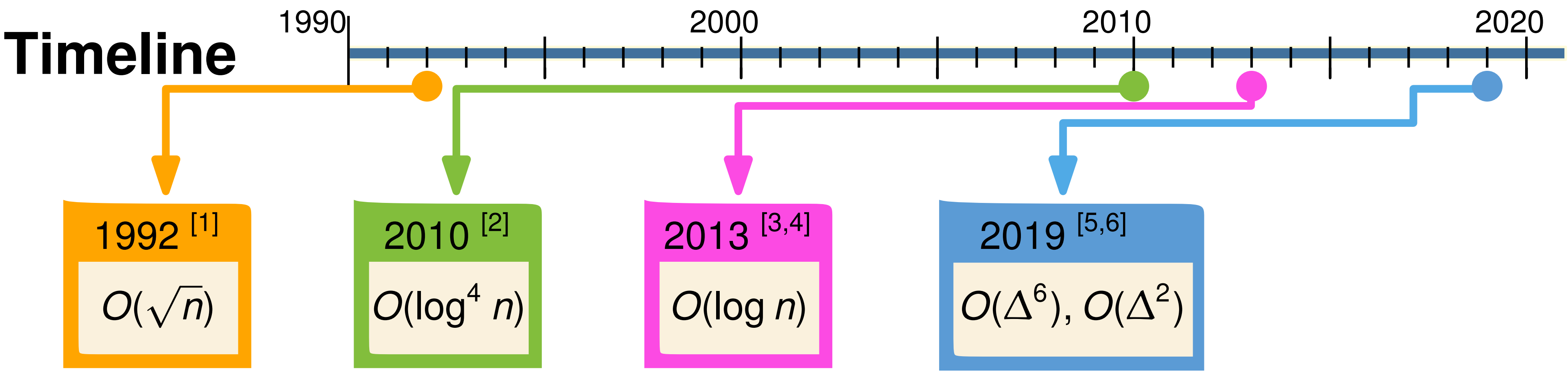

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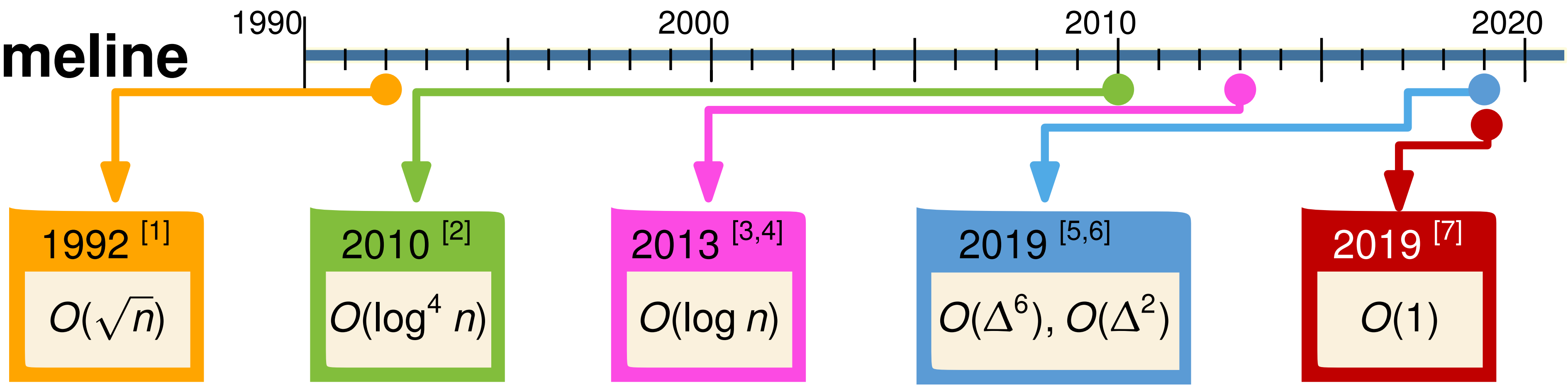

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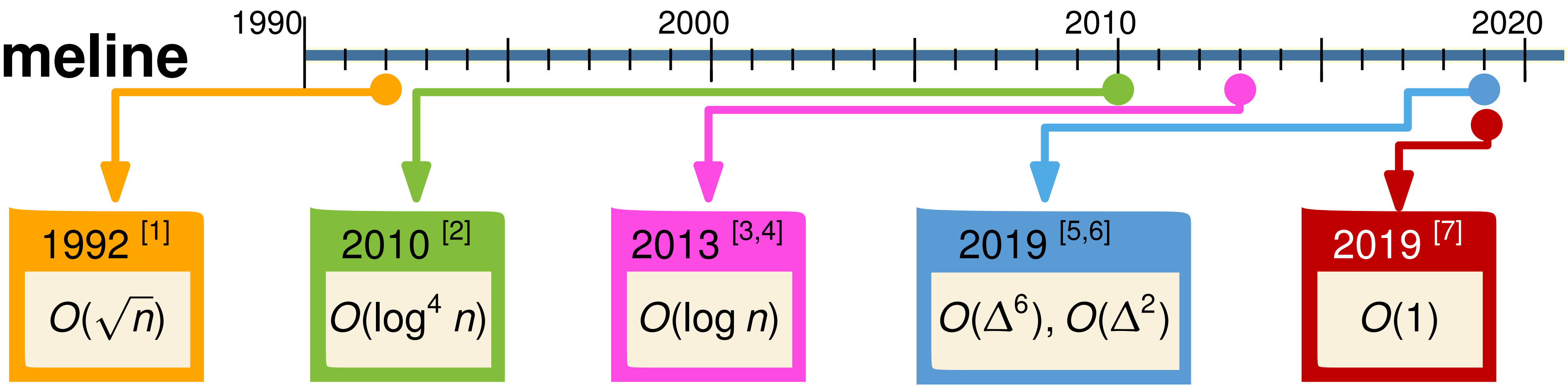

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Strong Product

Definition: $G \boxtimes H$

strong product of G and H

(assume that $G = P_k$)

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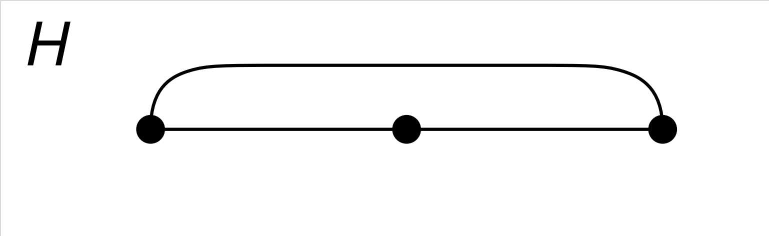

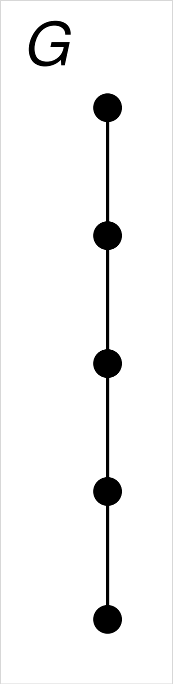

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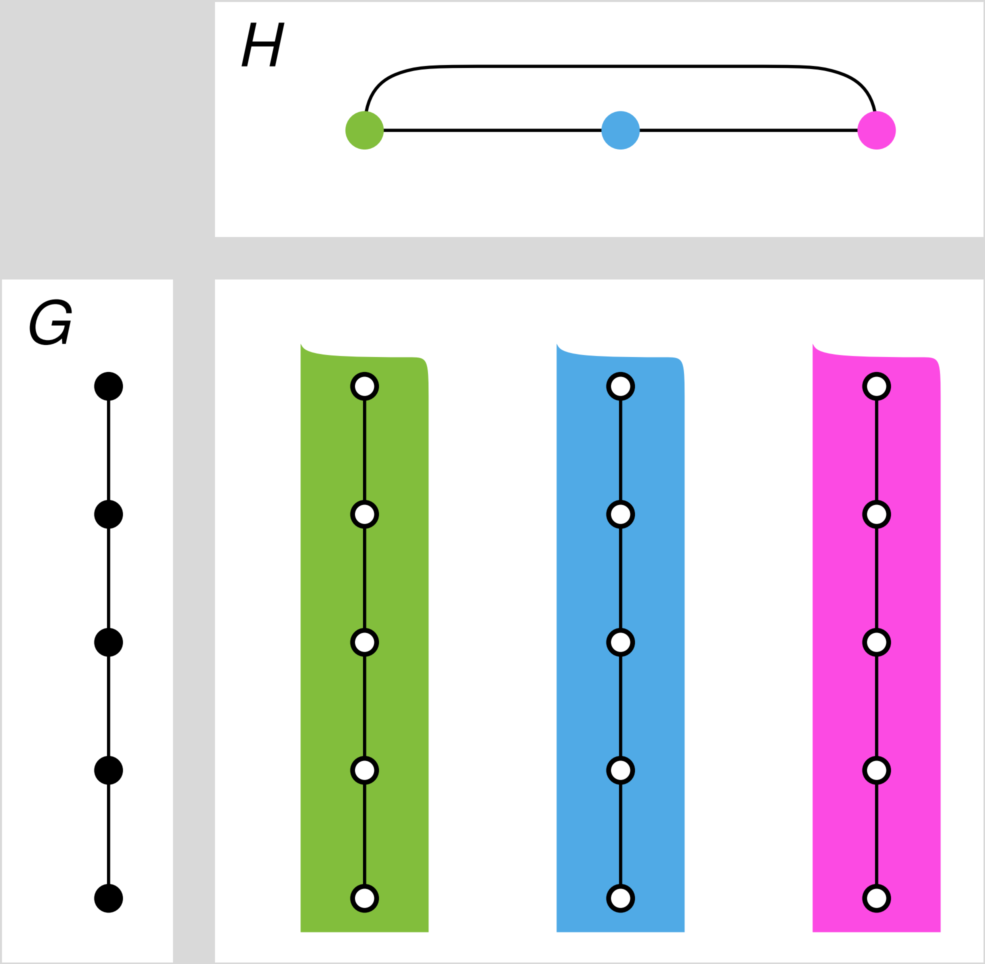
Definition: $G \boxtimes H$

strong product of G and H

(assume that $G = P_k$)

- vertices of H become copies of G



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- vertices of H become copies of G
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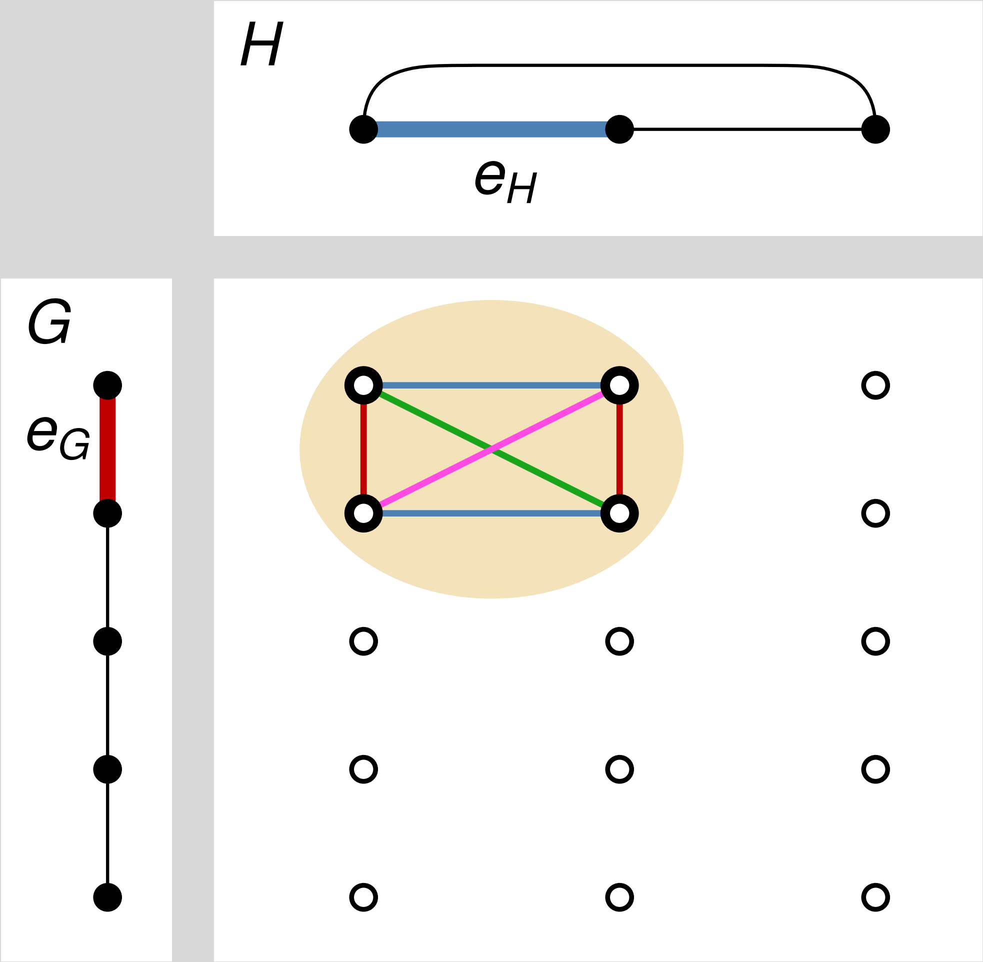

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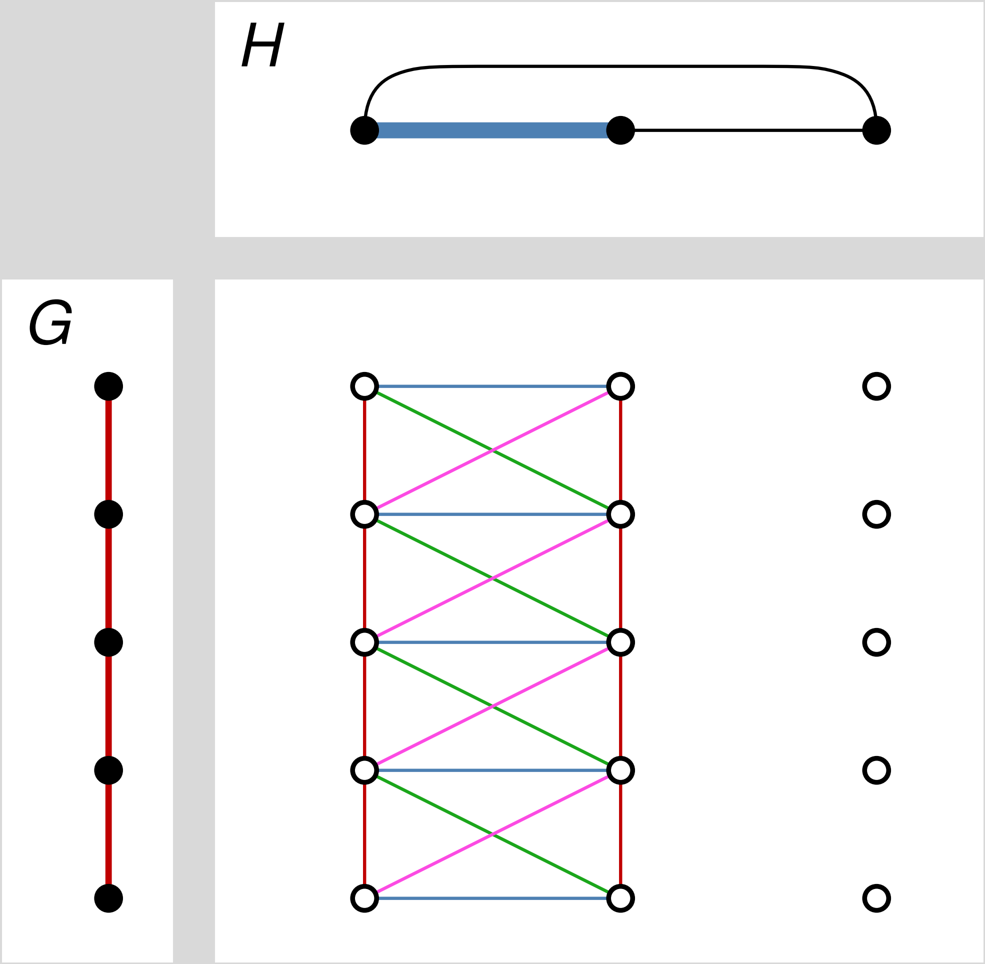

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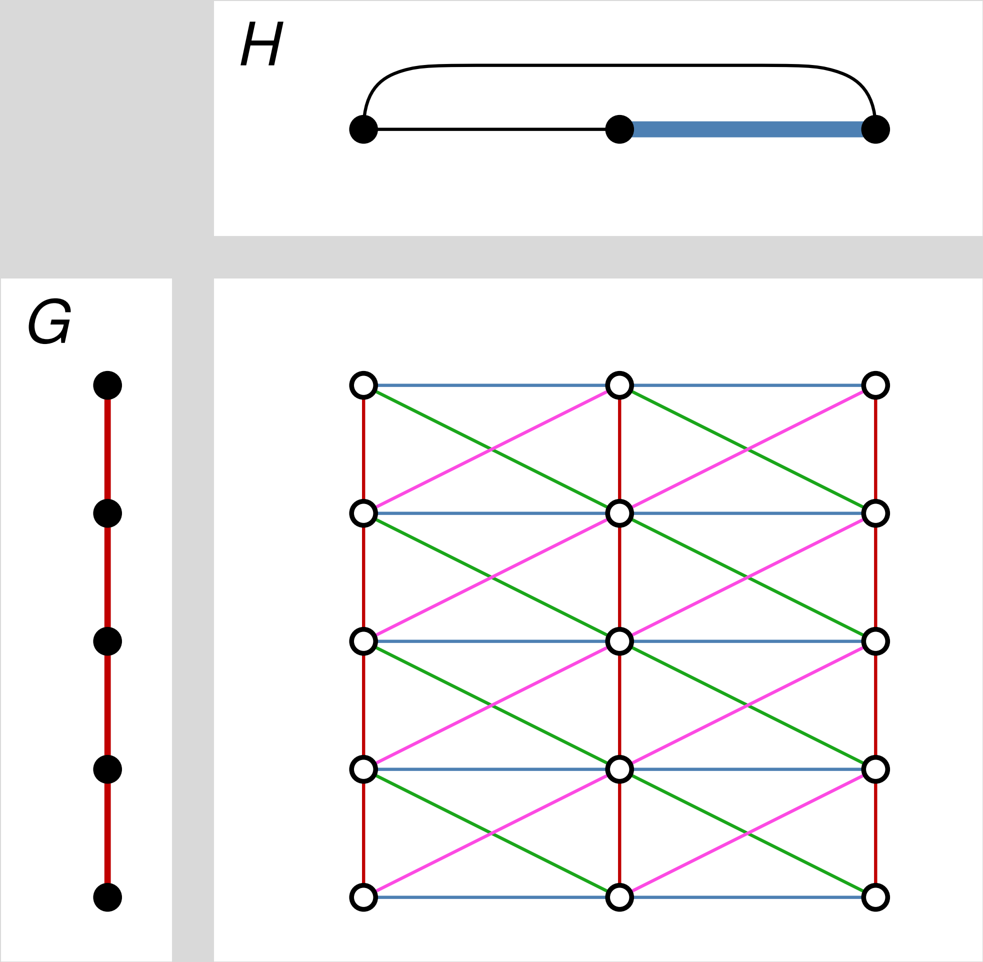

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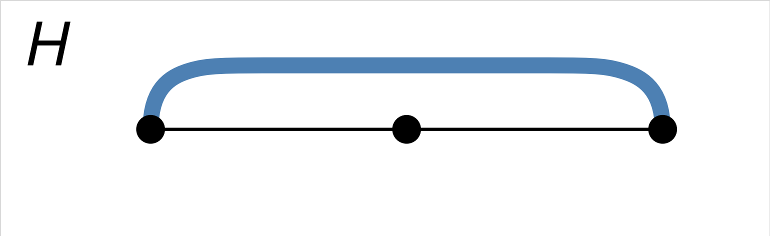

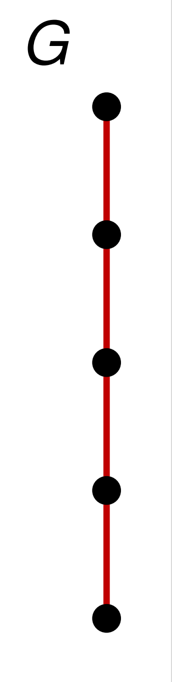

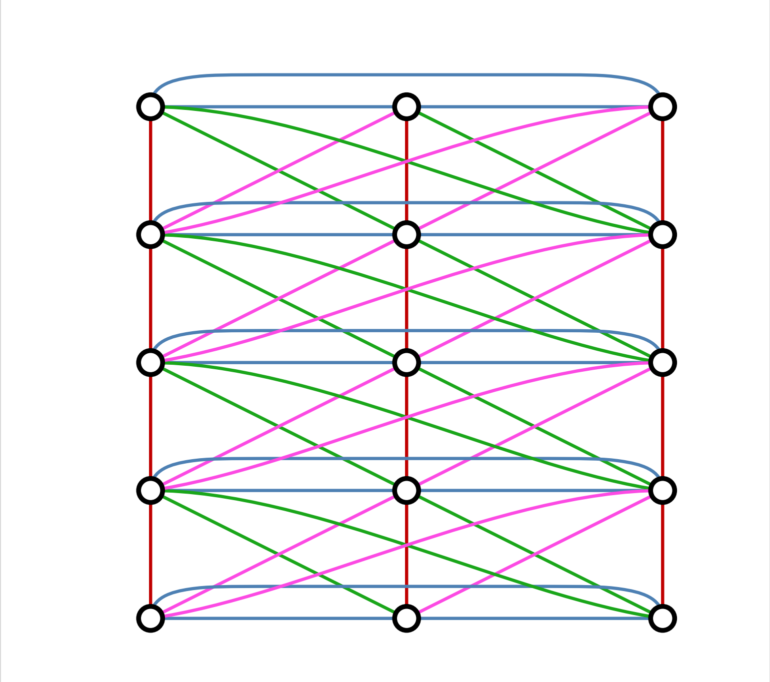

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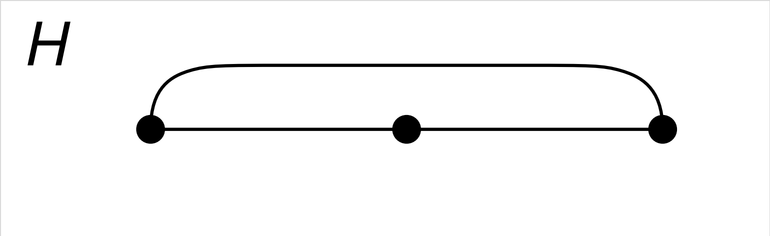
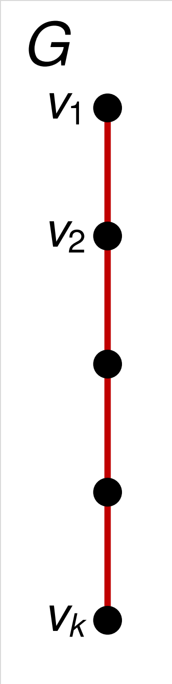
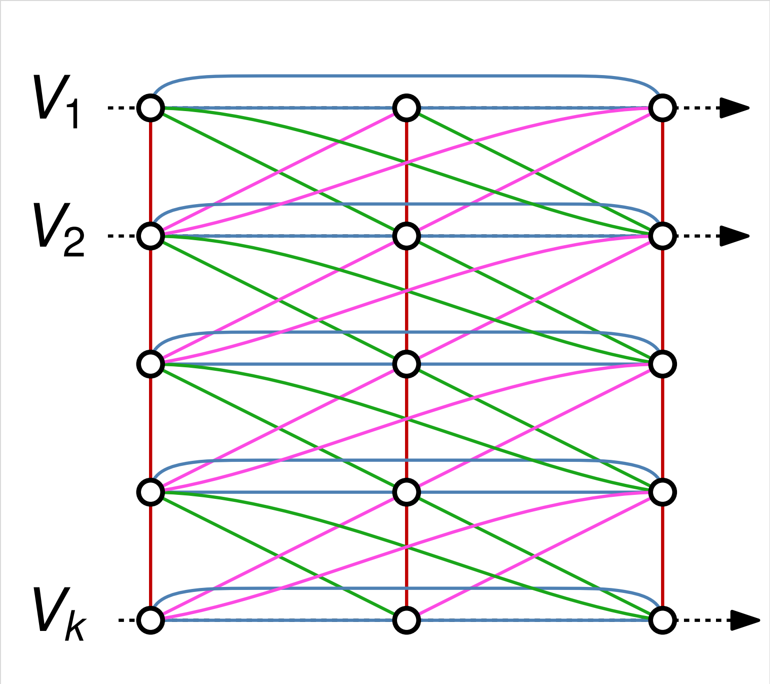
P_k implies a leveling $\{V_1, V_2, \ldots, V_k\}$







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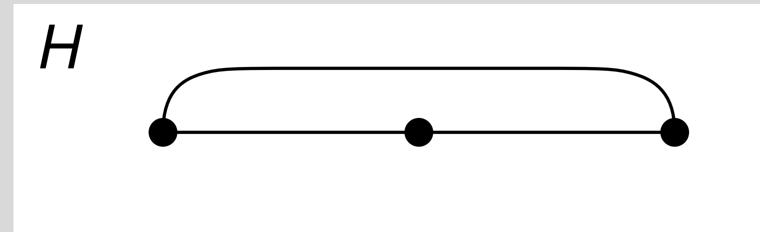
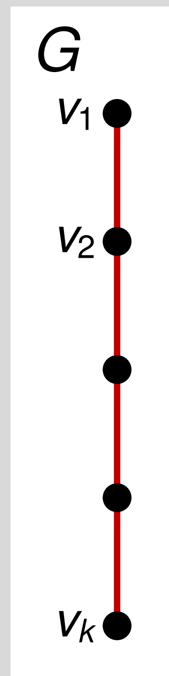
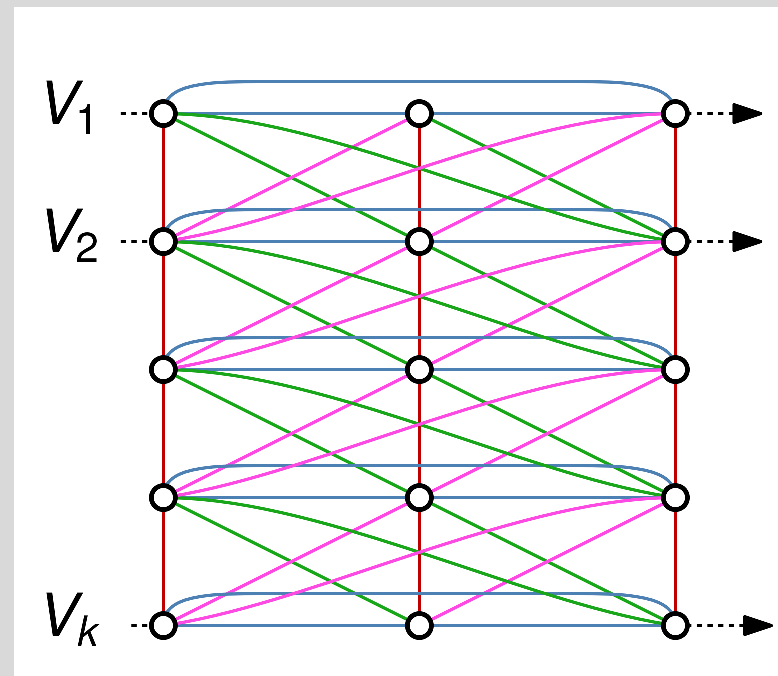
Types of edges







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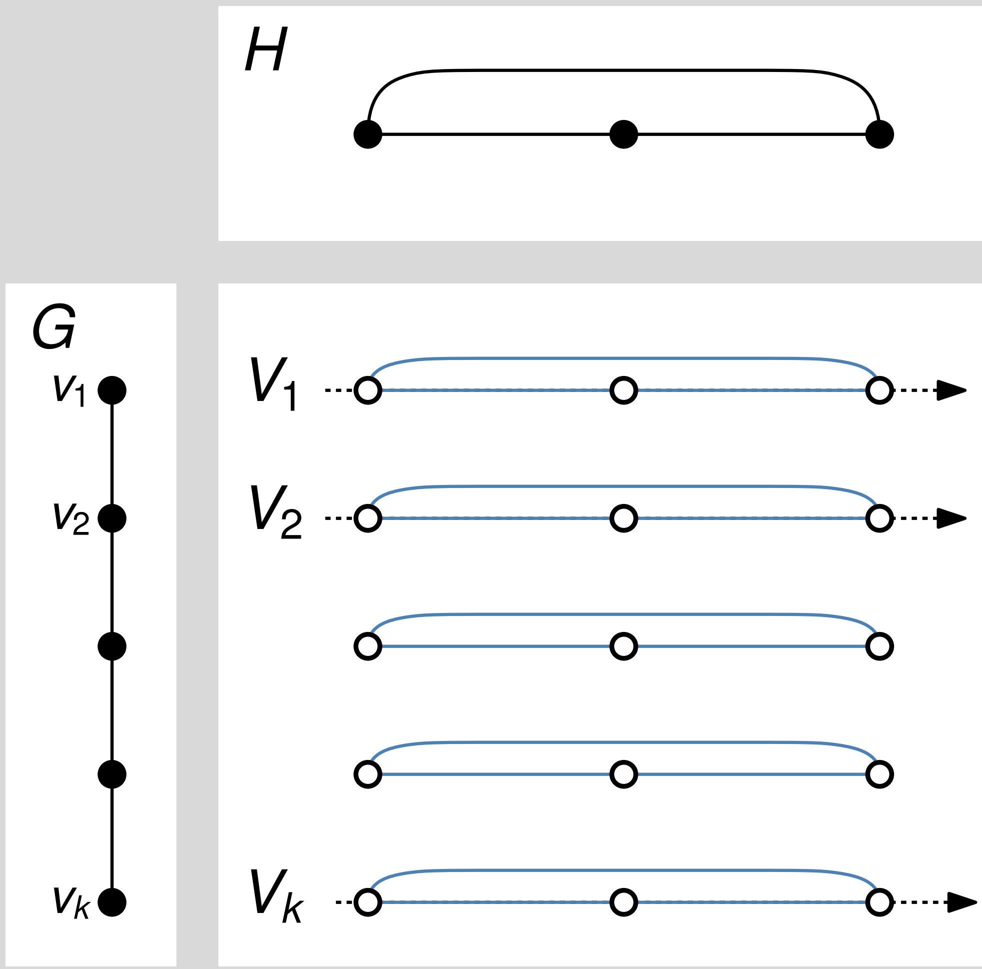
Types of edges

level



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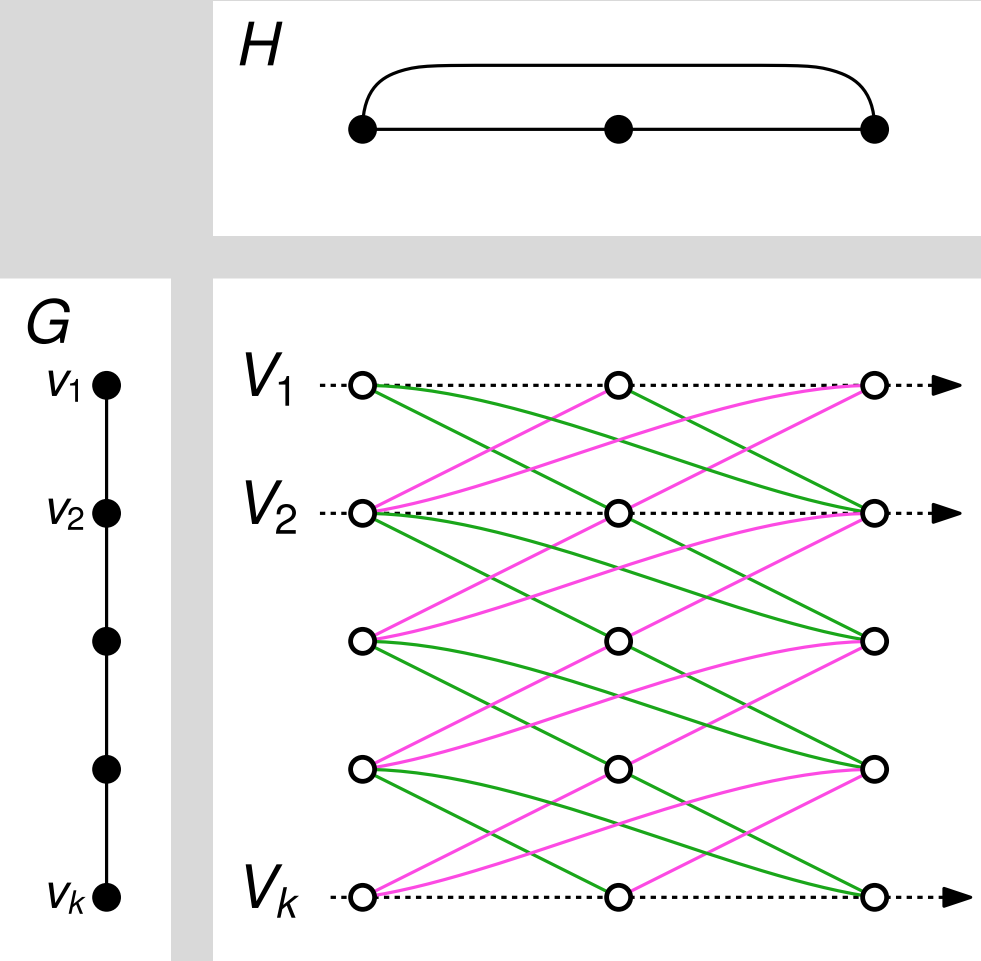
Types of edges

level

binding



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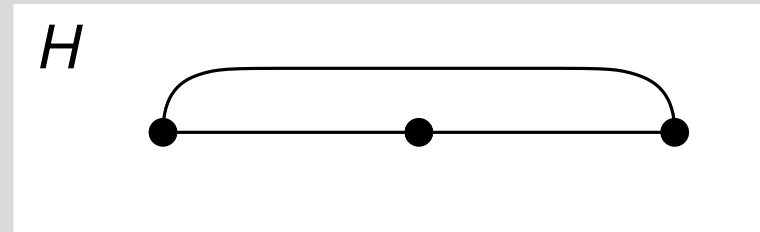
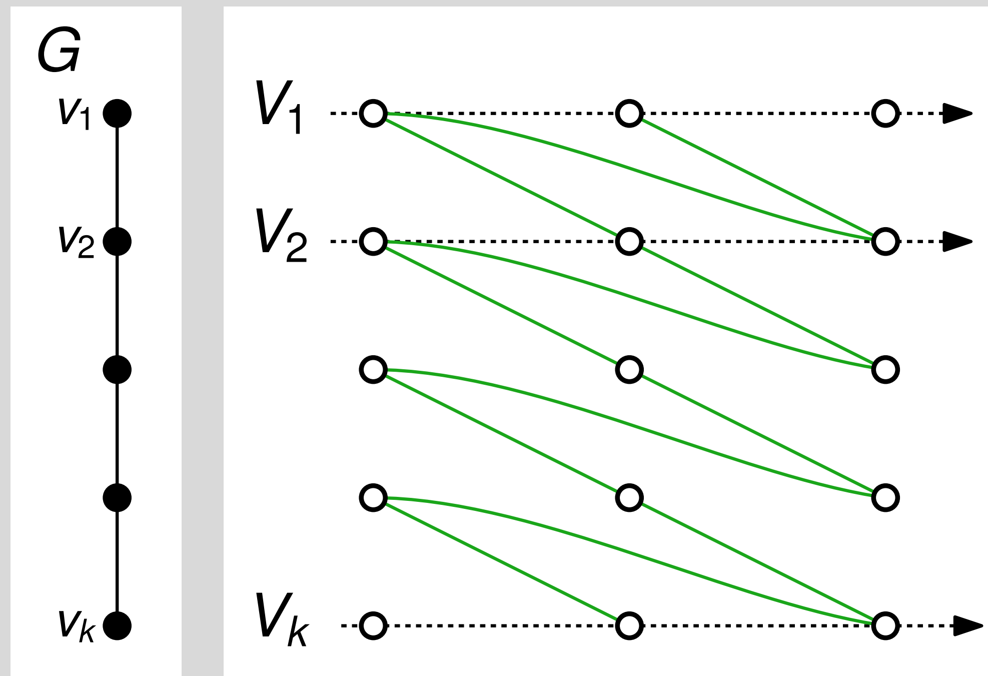
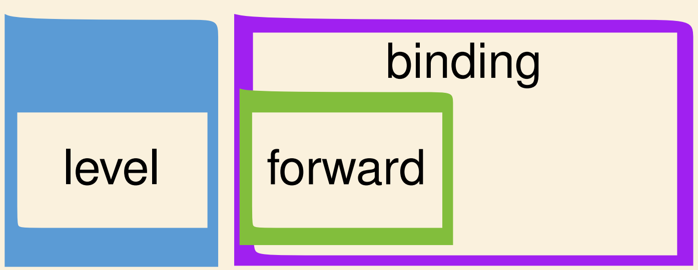




Types of edges



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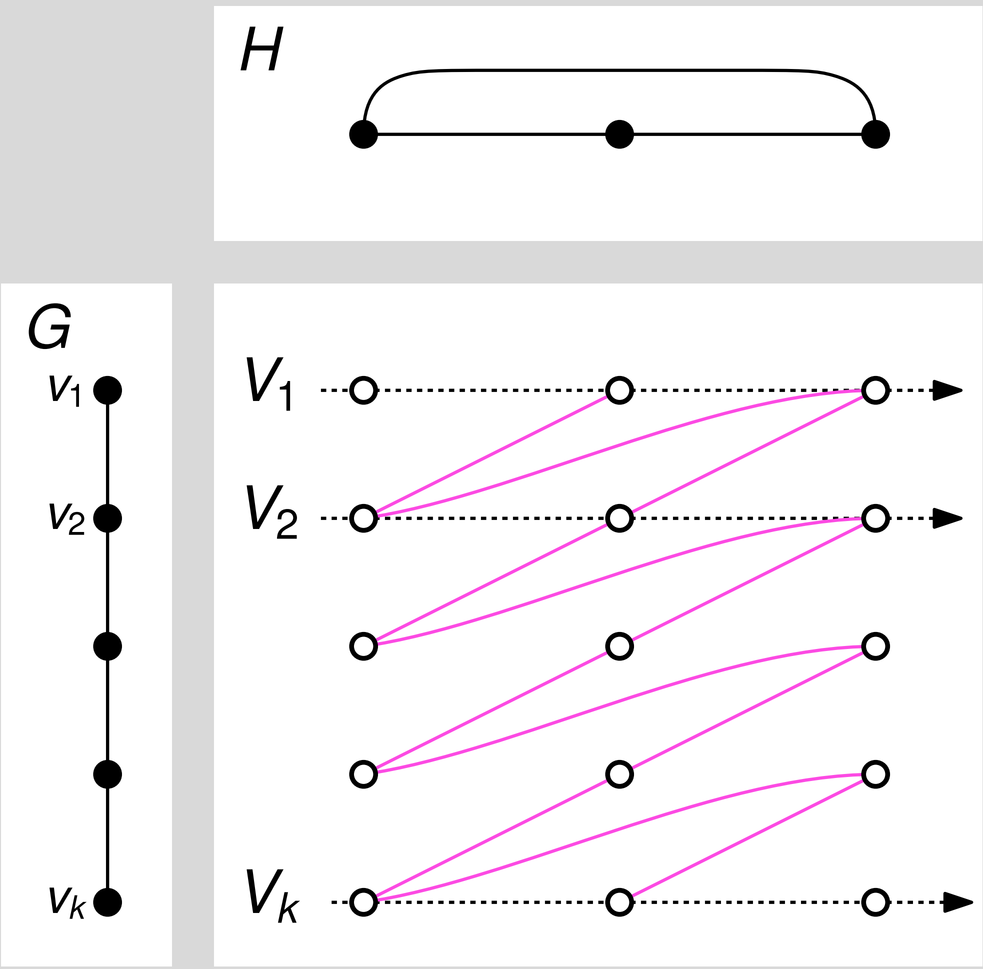
level

binding: forward, backward



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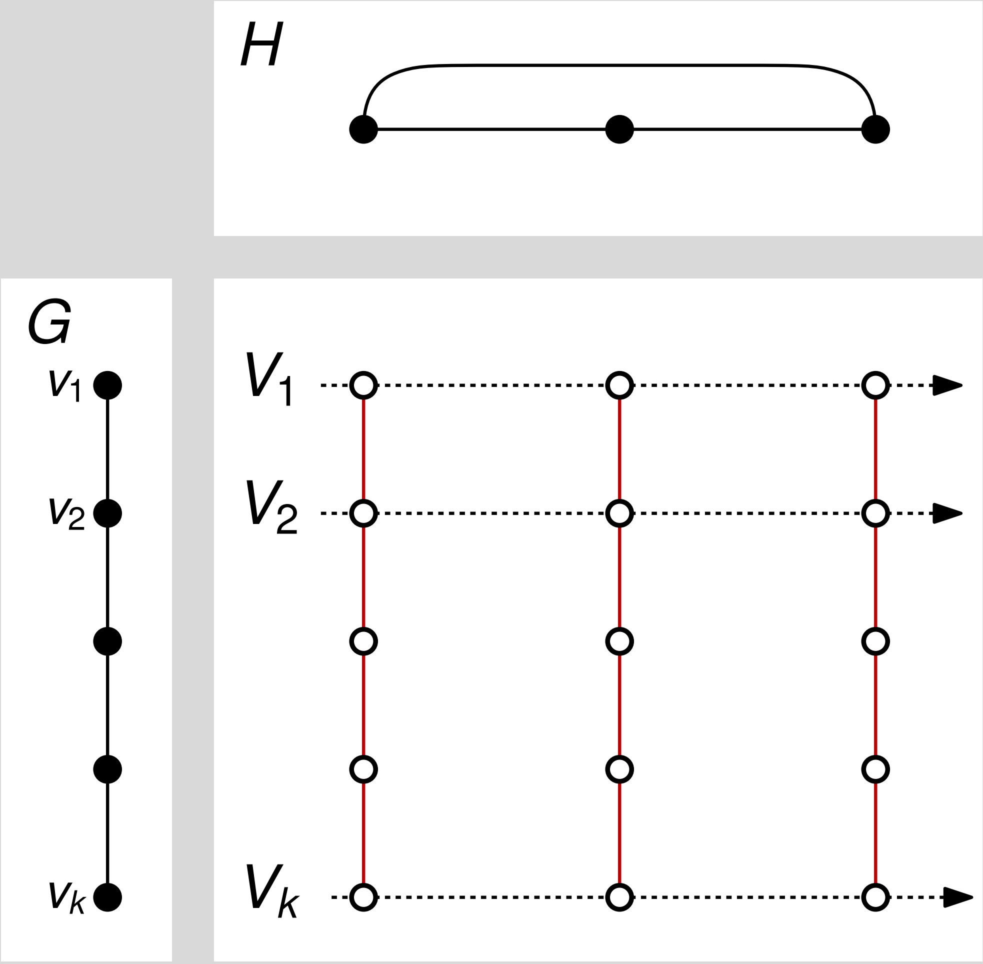

Types of edges

level

binding: forward, backward

vertex

Strong Product

Queue number[1]: $P \boxtimes H$

$$\mathrm{qn}(P \boxtimes H) \leq 3 \cdot \mathrm{qn}(H) + 1$$

[1] Wood, *Queue layouts of graph products and powers* (2005)

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Theorem [2]:

$\text{qn}(G) = k$ if and only if there is a vertex order with no $(k+1)$-rainbow

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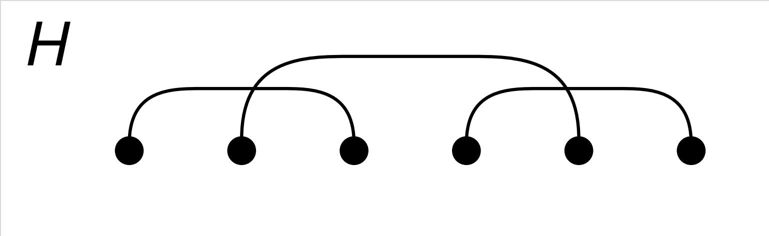
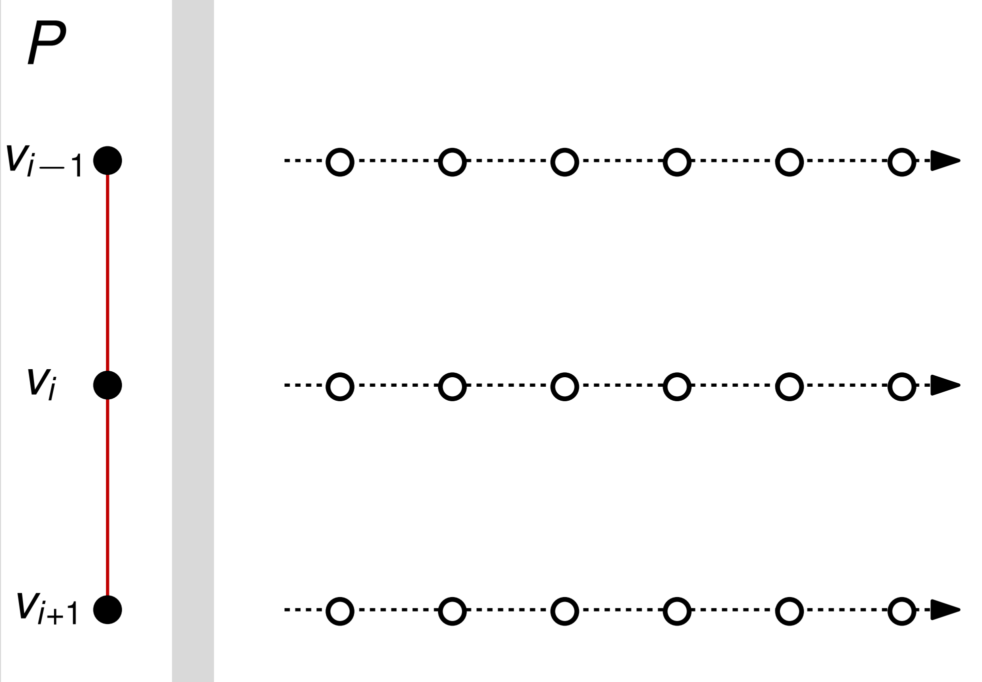
- queue layout of H with $\text{qn}(H)$ pages

edges on one queue of H





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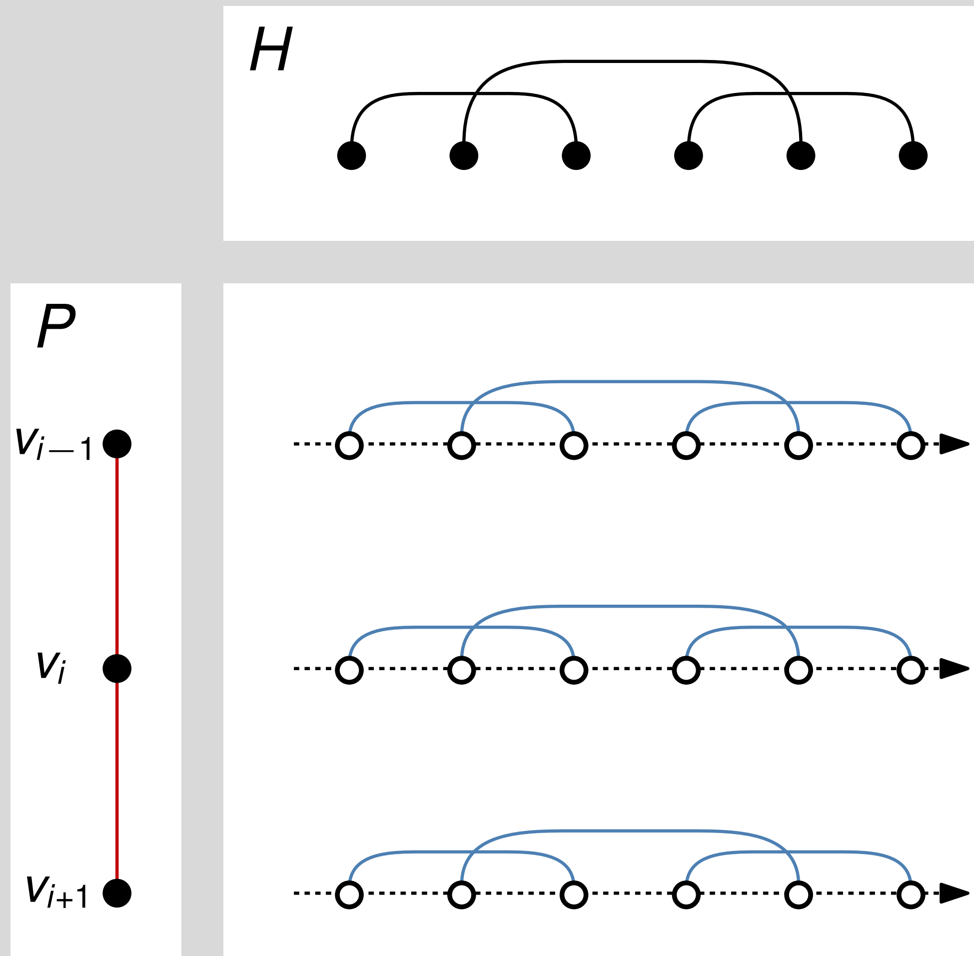
level

one



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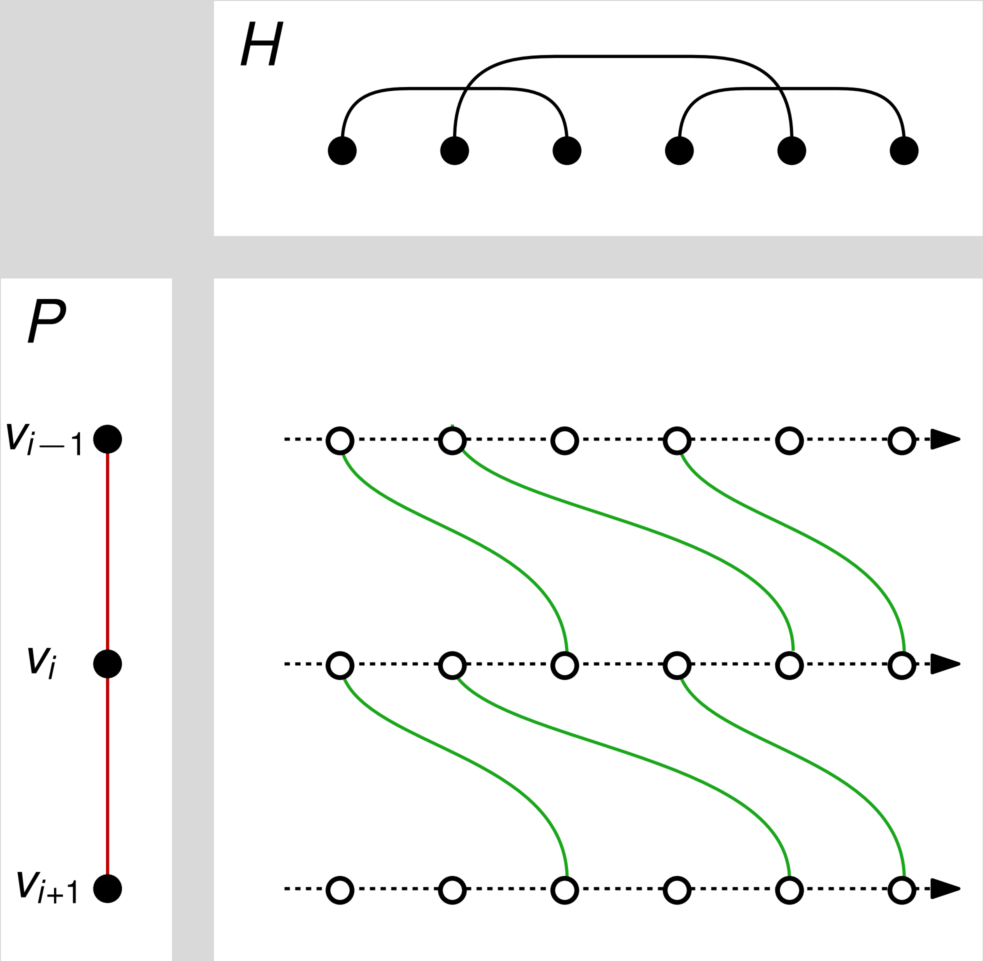
edges on one queue of H

level	forward
one	one



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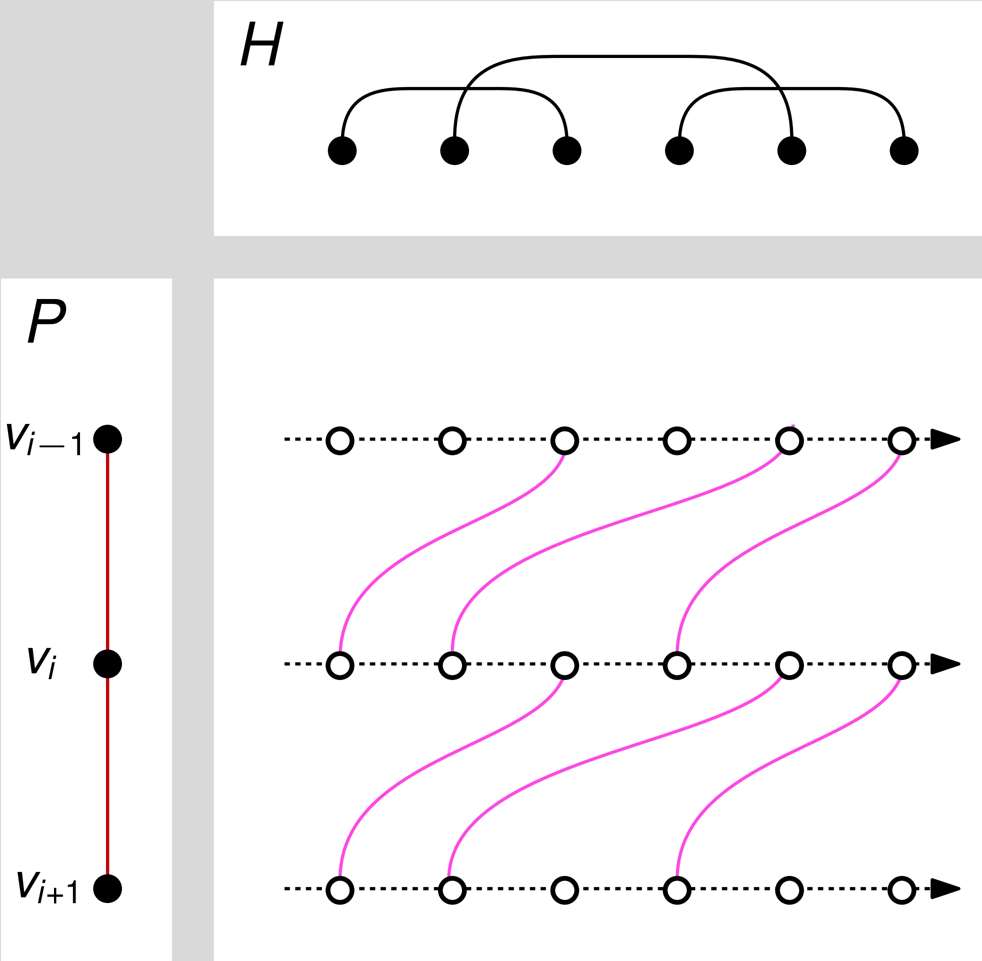
level	forward	backward
one	one	one



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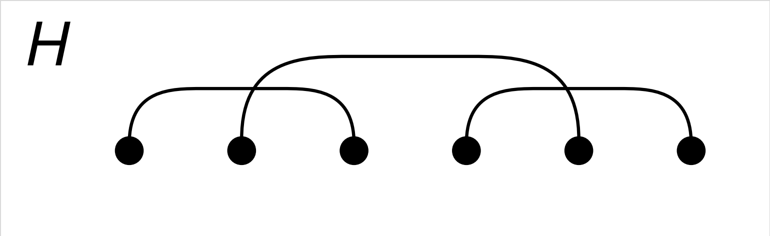
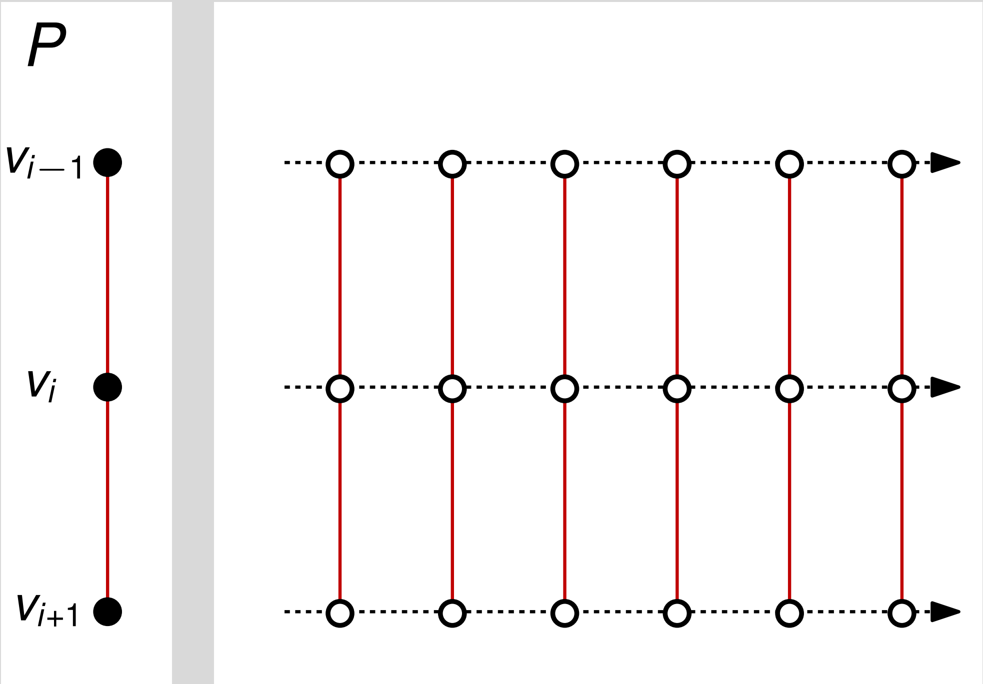
one	one	one

vertex
one





[1] Wood, *Queue layouts of graph products and powers* (2005)
[2] Heath, Rosenberg: *Laying out graphs using queues* (1992)

Strong Product

Queue number[1]: $P \boxtimes H$

$$\mathrm{qn}(P \boxtimes H) \leq 3 \cdot \mathrm{qn}(H) + 1$$

Theorem [2]:

qn(G) = k if and only if there is a vertex order with no ($k + 1$)-rainbow

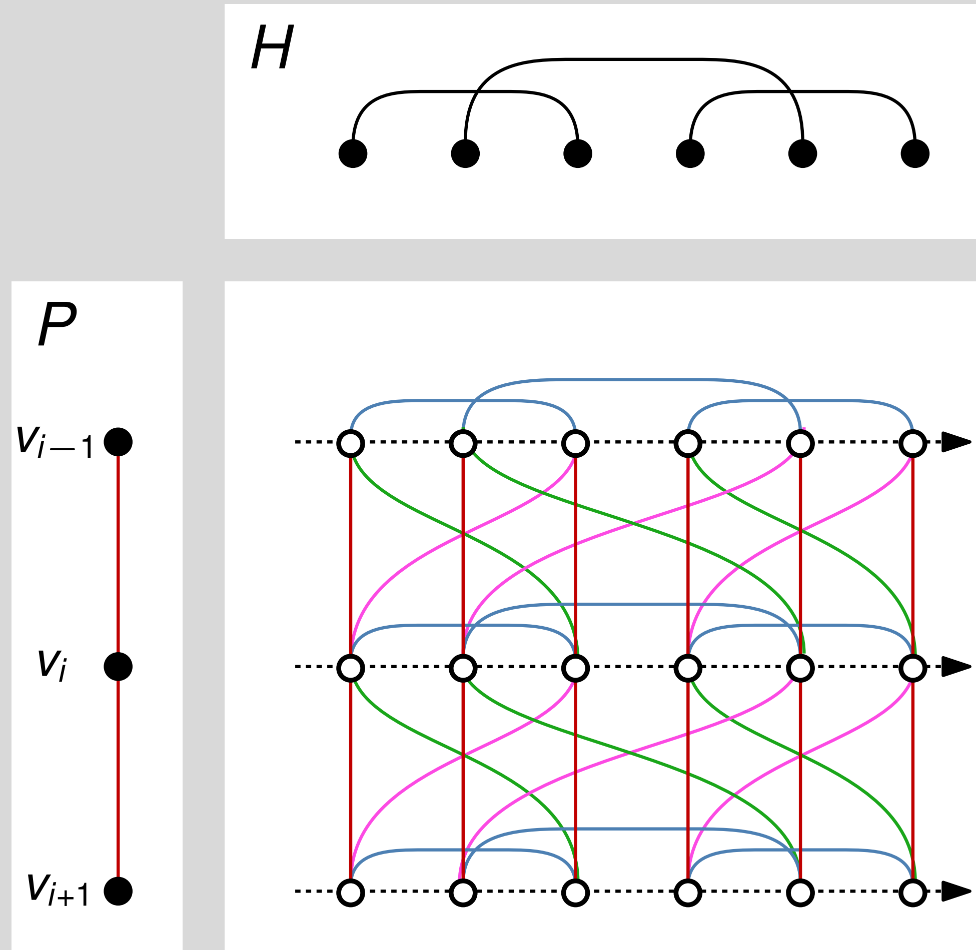
- queue layout of H with qn(H) pages

edges on one queue of H

level	forward	backward
one	one	one

vertex
one

$\boxed{3} \cdot \mathrm{qn}(H) + \boxed{1}$



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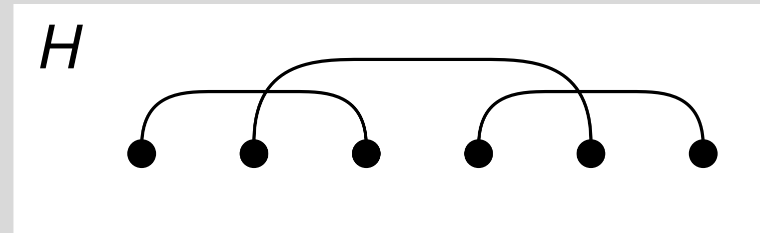

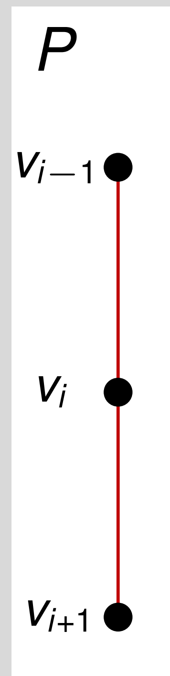

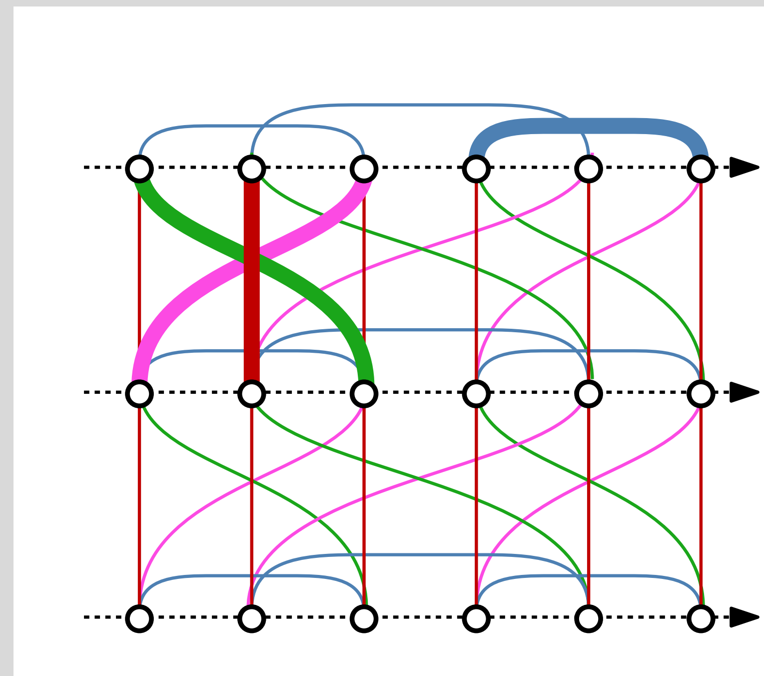

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Strong Product of planar graphs

Theorem[1]:

G planar subgraph of $P_k \boxtimes H \boxtimes K_3$

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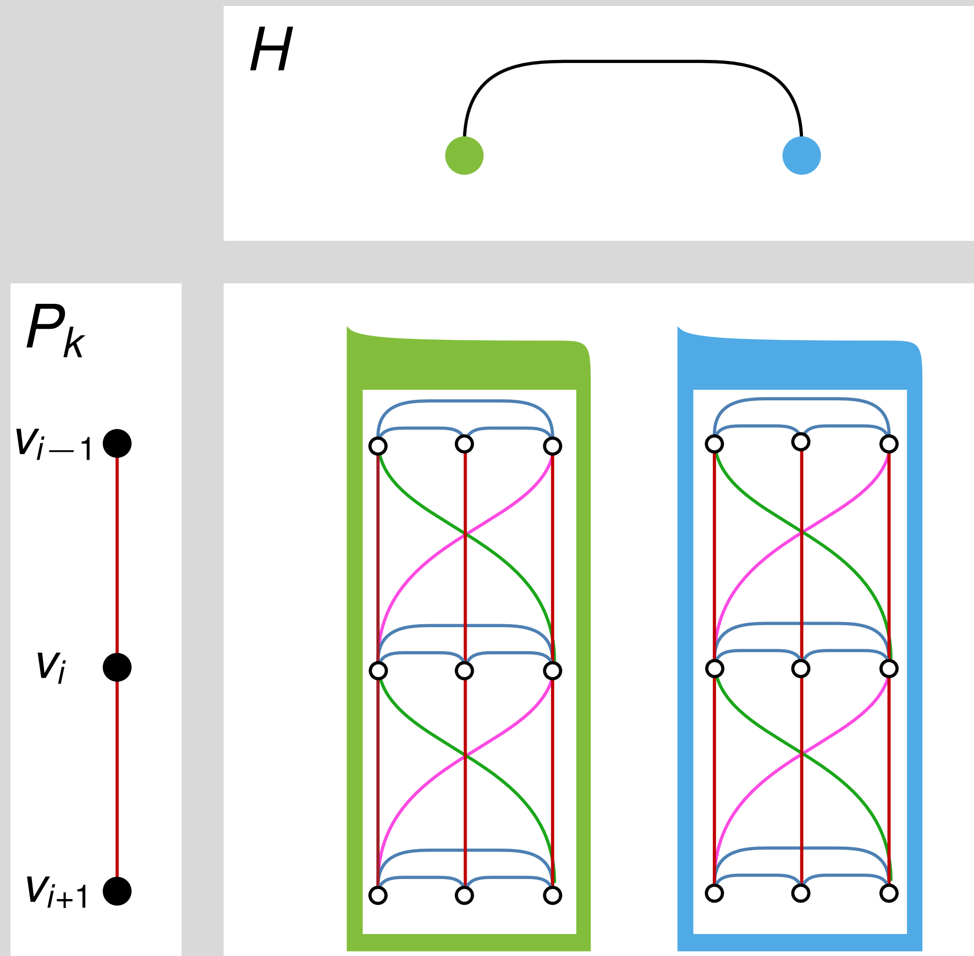

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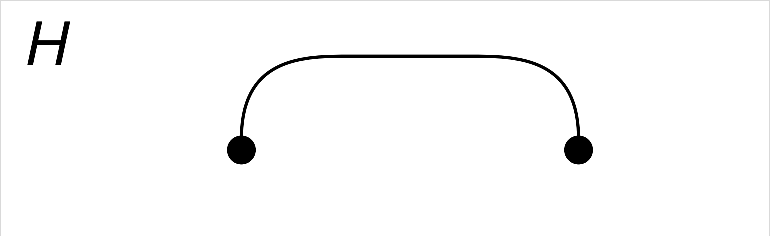

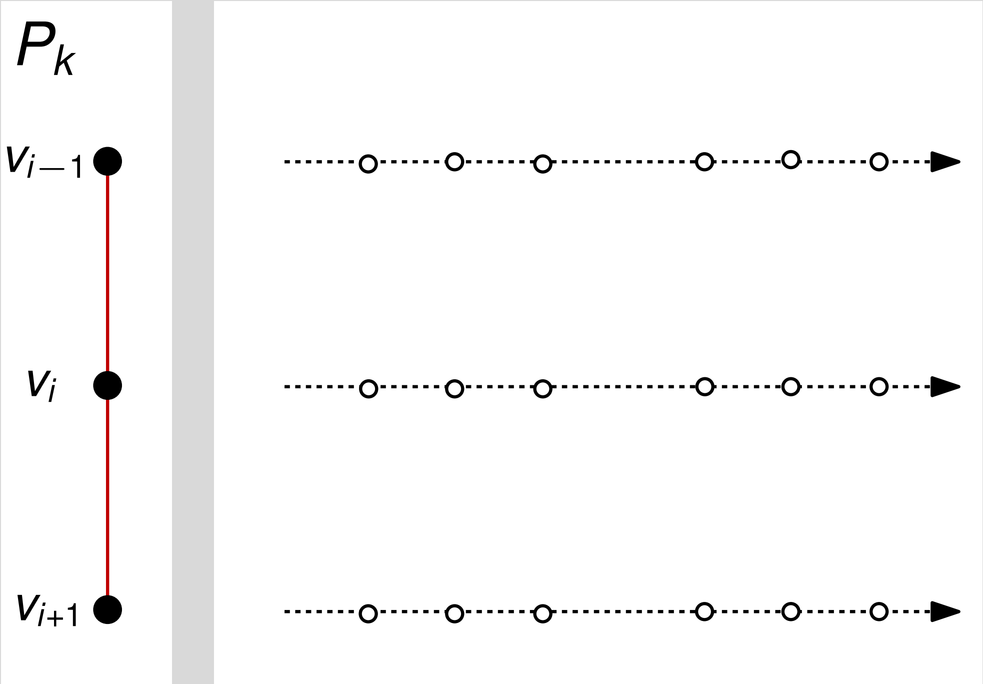

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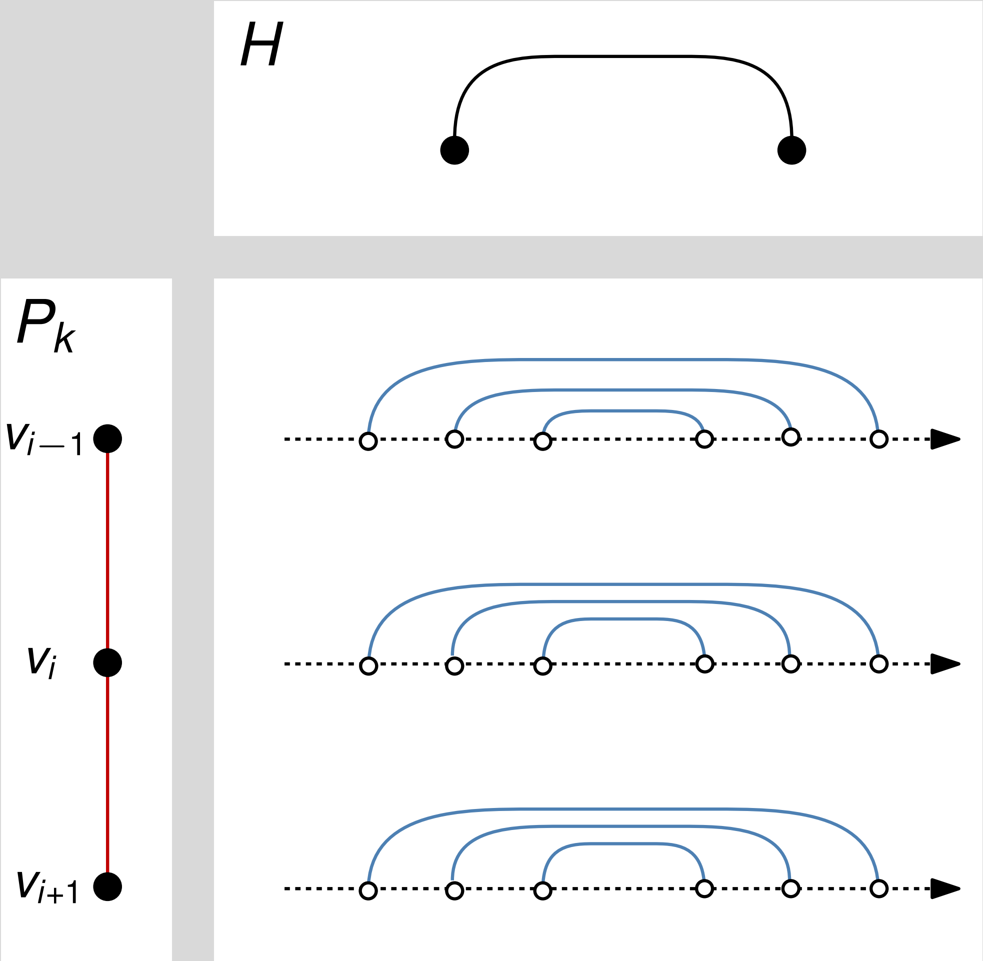
edges on one queue of H

level

three



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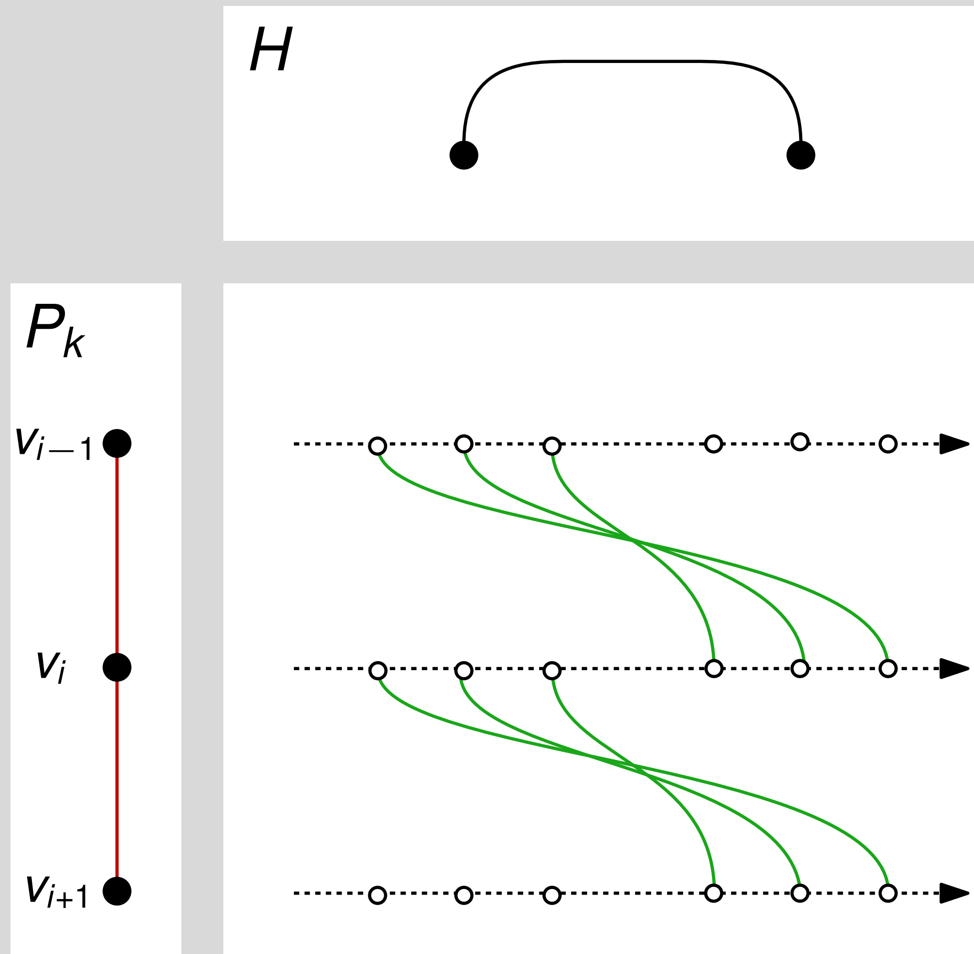
edges on one queue of H

level	forward
three	three



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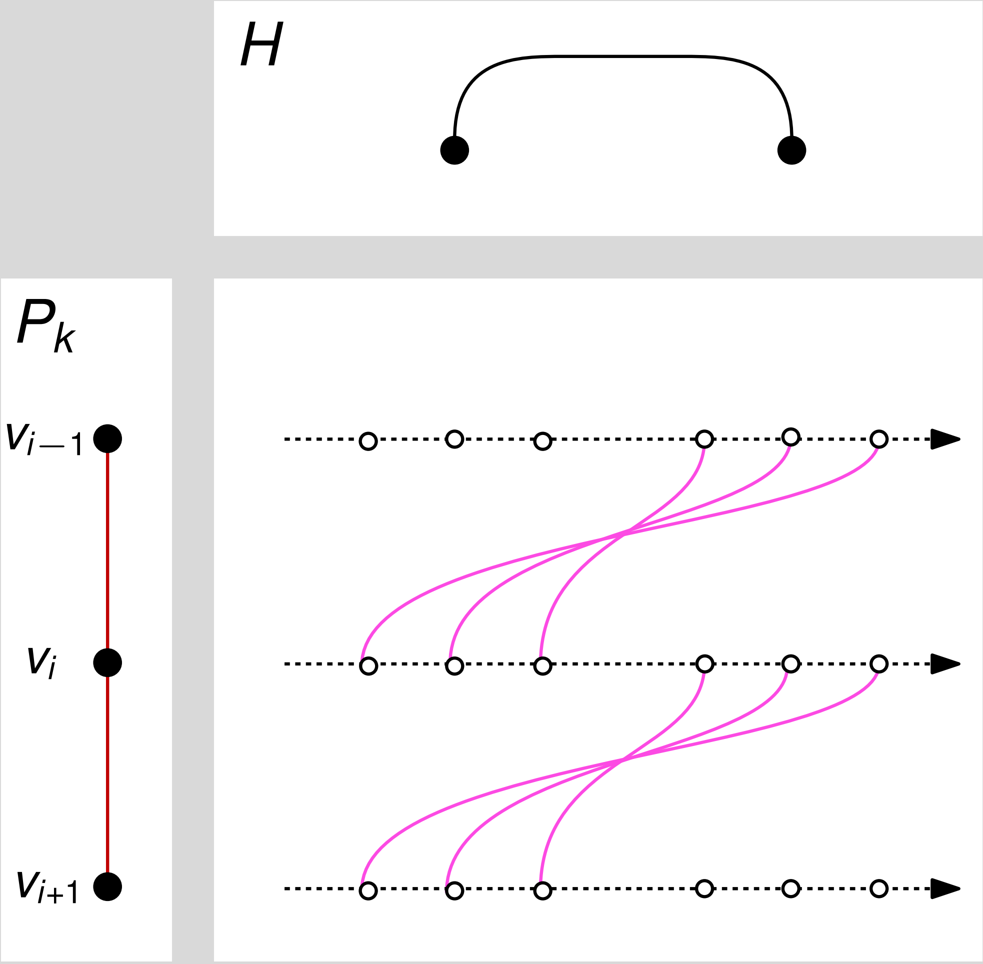
level	forward	backward
three	three	three



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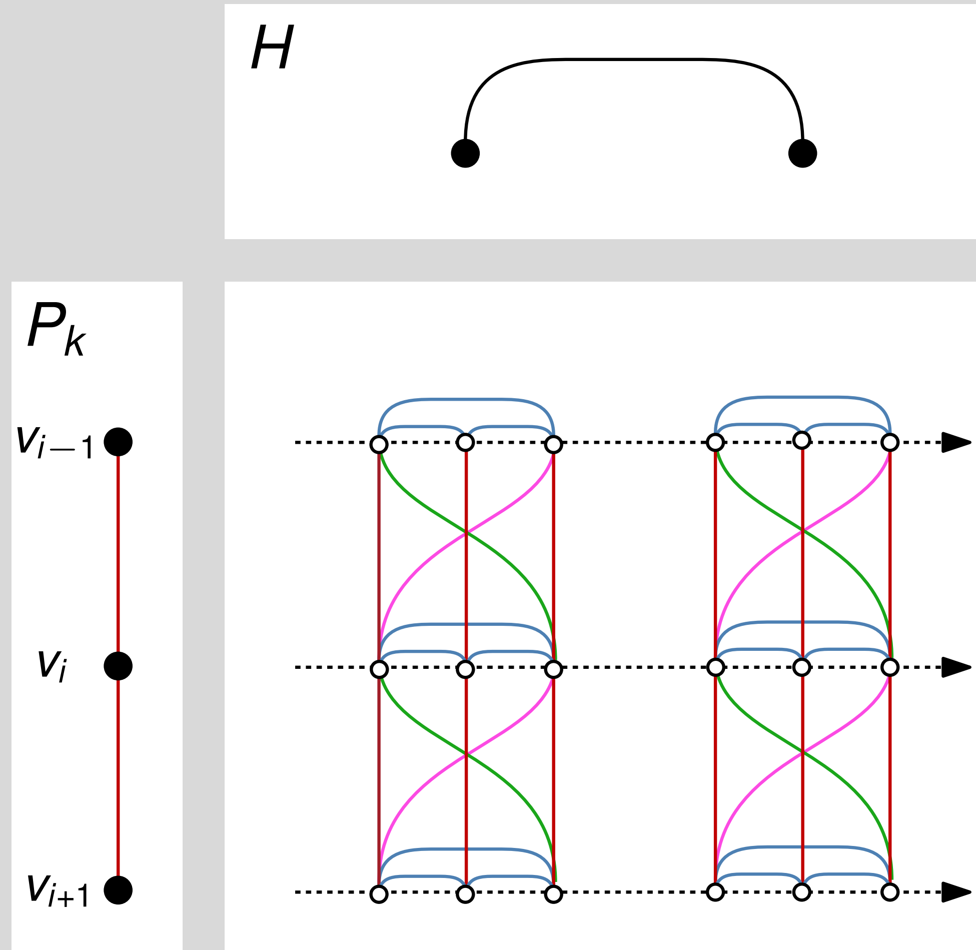
level	forward	backward
three	three	three

vertex

$3qn(K_3) + 1$



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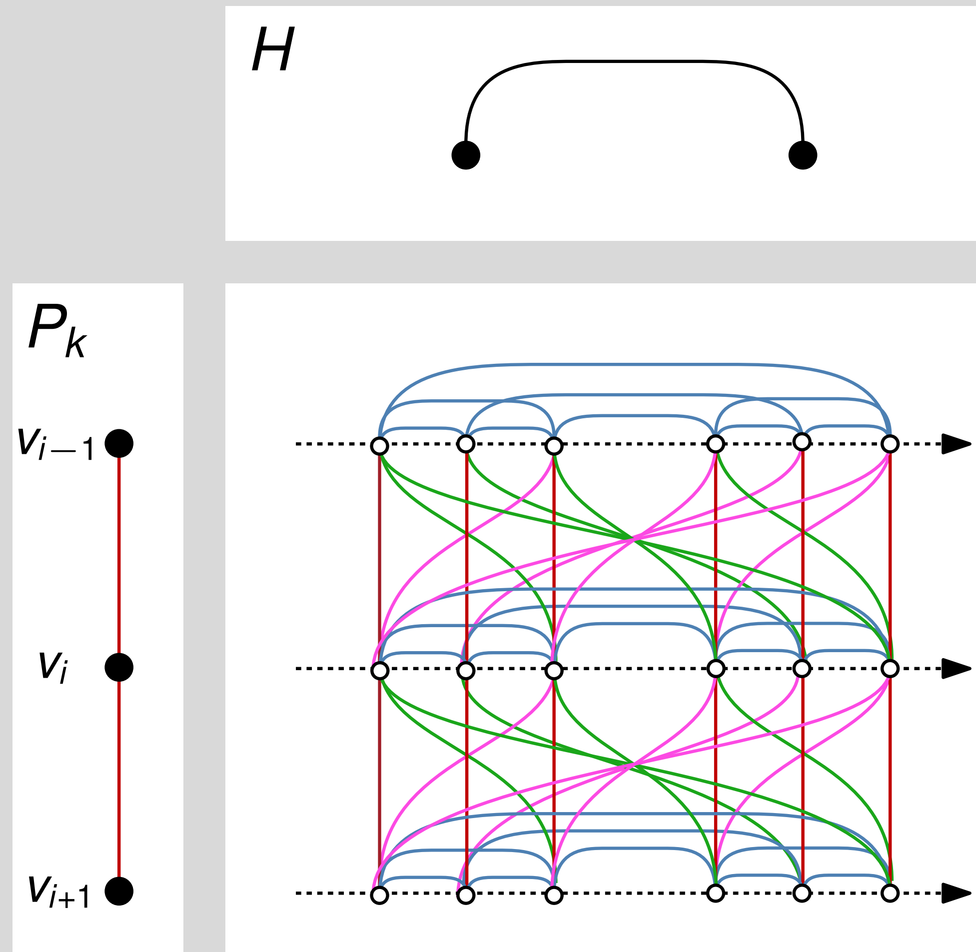
edges on one queue of H

level	forward	backward
three	three	three

vertex: $3\text{qn}(K_3) + 1$

$3 \cdot 3 \cdot \text{qn}(H) + 4$



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Step 1: compute graph H

Theorem [Dujmović et al.]:

Planar graph G subgraph of $P \boxtimes H \boxtimes K_3$ where H is a planar 3-tree

$$\mathrm{qn}(G) \leq 3 \cdot 3 \cdot \mathrm{qn}(H) + 4 \leq 49 \qquad (\mathrm{qn}(H) \leq 5)$$

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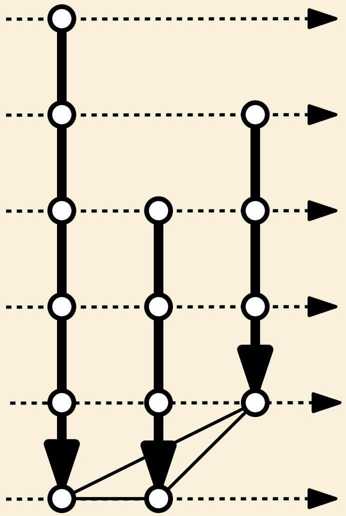

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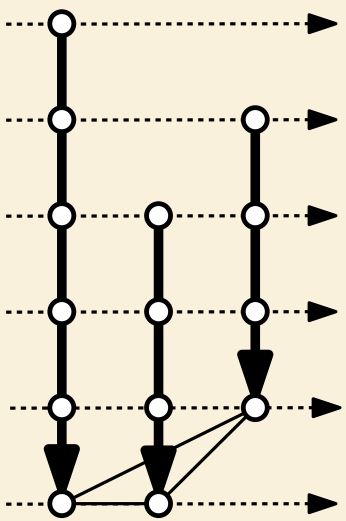

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Tripod decomposition

- vertex partition in tripods

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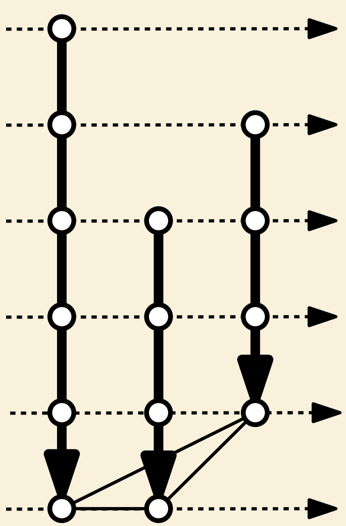

subgraph of $P \boxtimes K_3$

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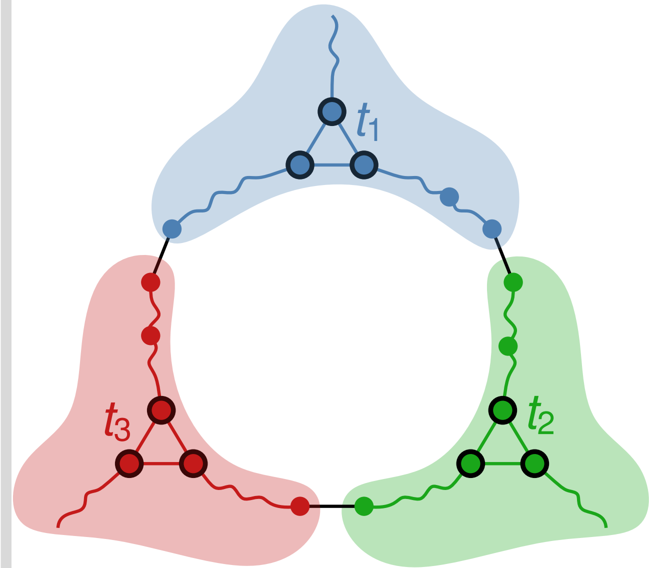
- vertex partition in tripods
- outerface bounded by t_1, t_2, t_3



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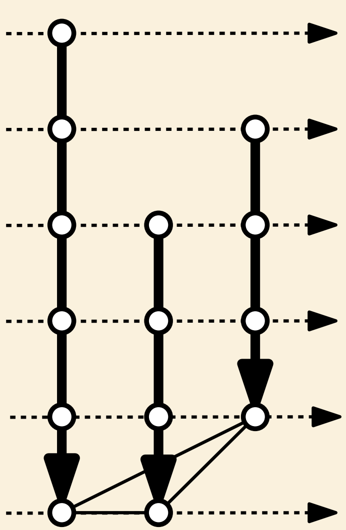

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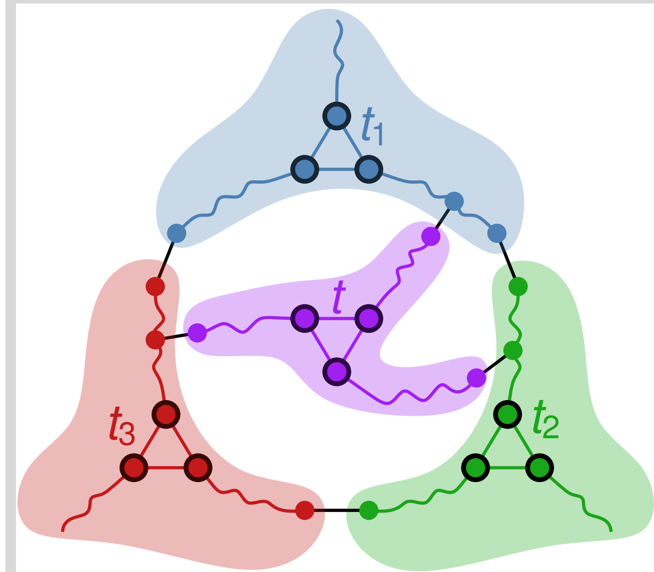

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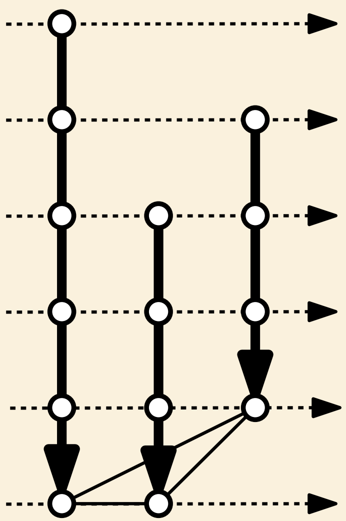

subgraph of $P \boxtimes K_3$

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- vertex partition in tripods
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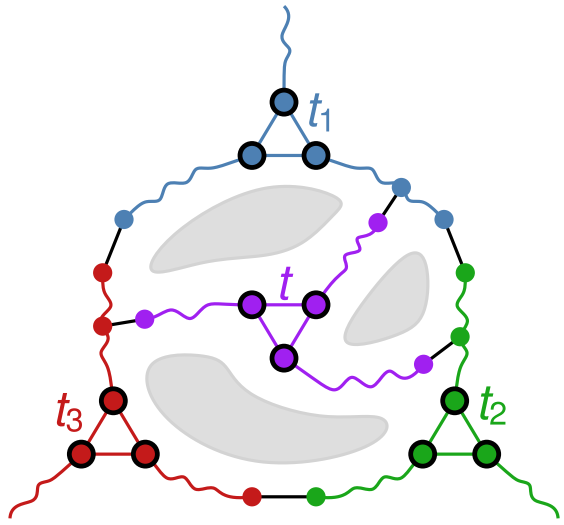
- recurse in regions



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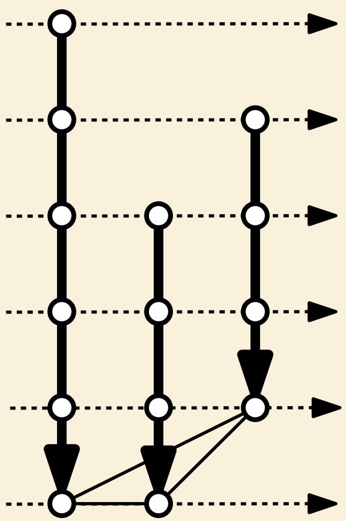

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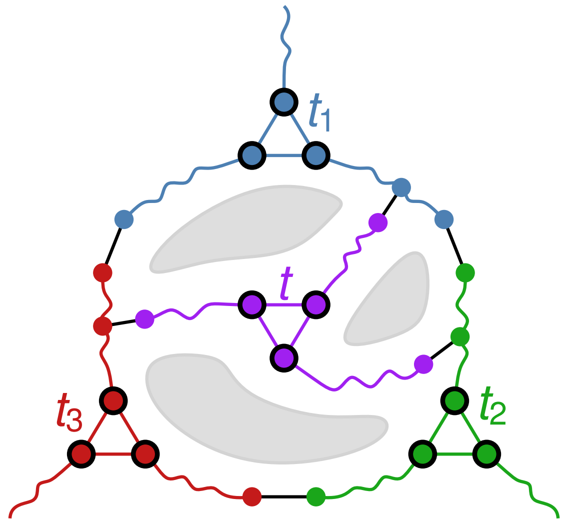
- t_1, t_2, t_3 parents of t



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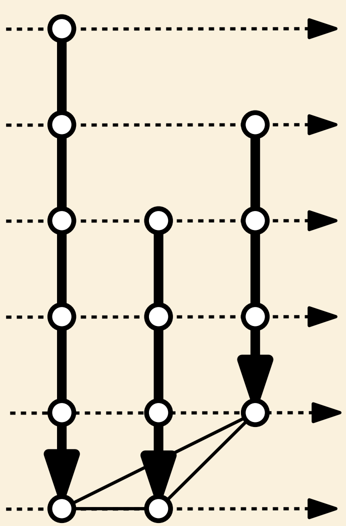

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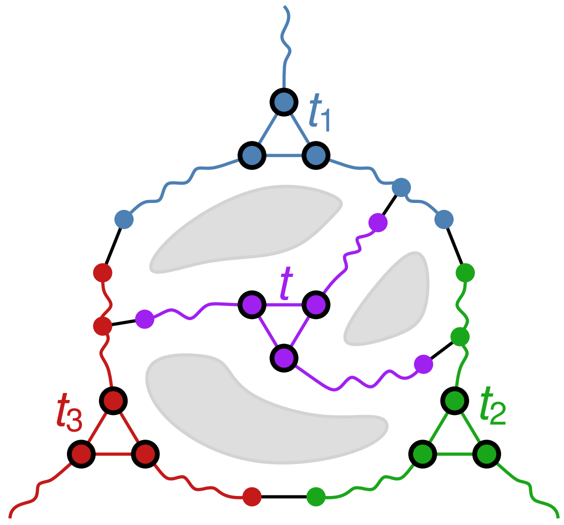

Graph H

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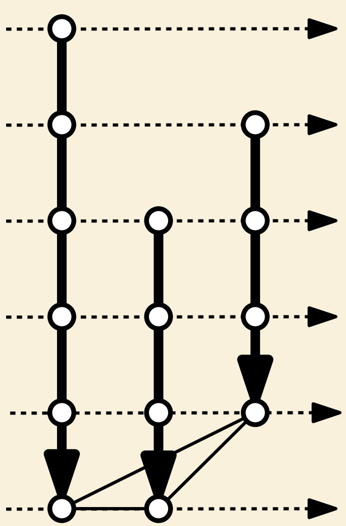

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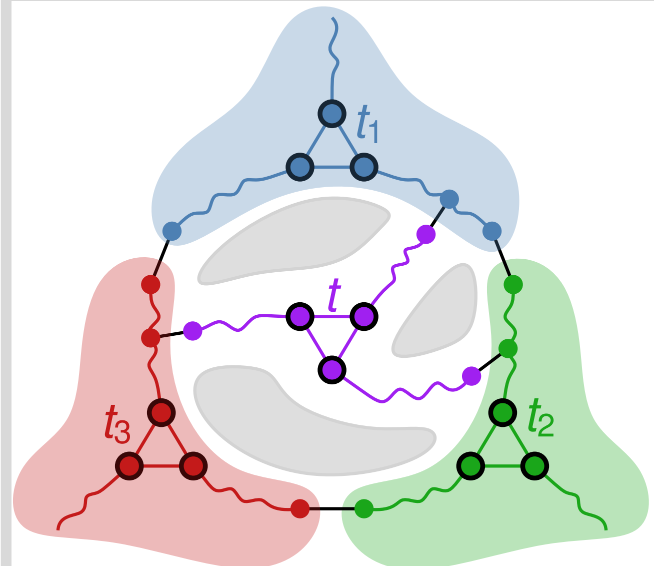

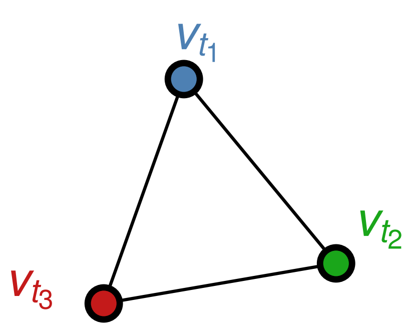

Graph H

- vertex v_t corresponds to tripod t

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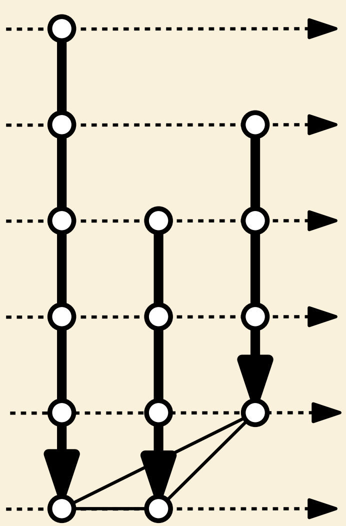

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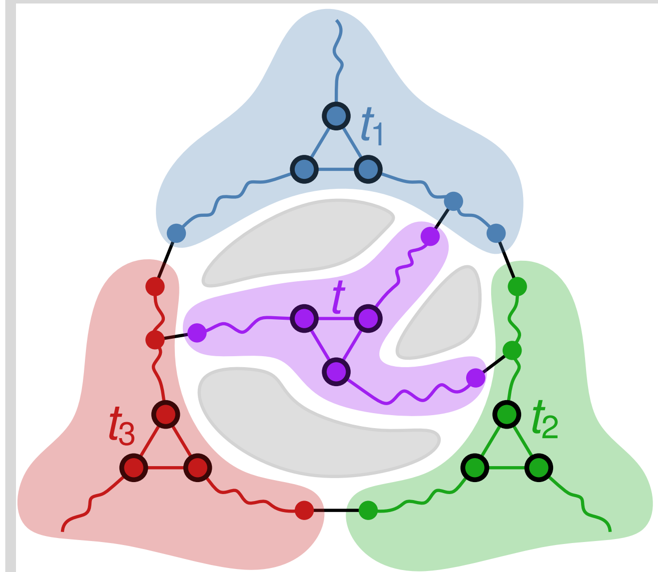

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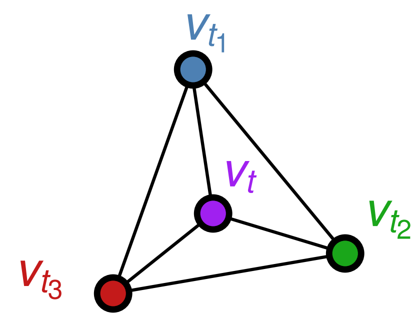

Graph H

- vertex v_t corresponds to tripod t
- v_t adjacent with v_{t_1}, v_{t_2}, v_{t_3}

Step 2: 5-queue layout of H

Theorem[1]:

A planar 3-tree H has $\text{qn}(H) \leq 5$

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Queue layouts of planar 3-trees (2020)

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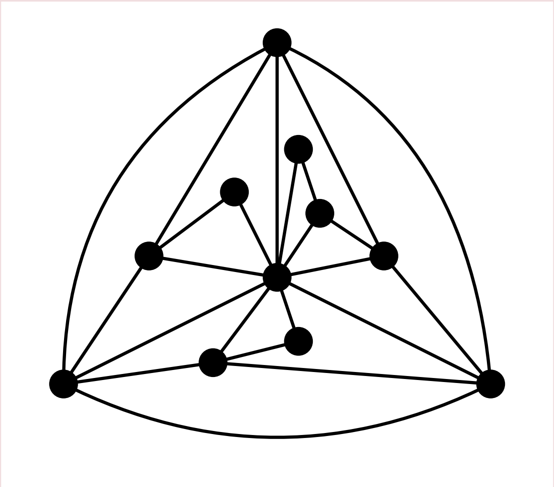

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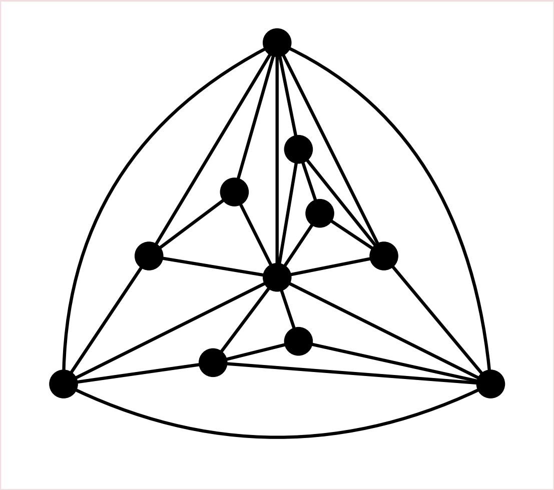

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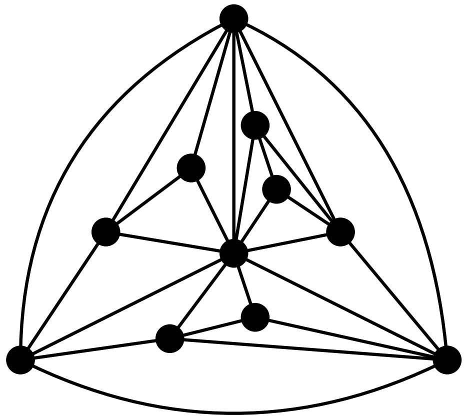
- maximal planar 3-tree
- leveling of vertices



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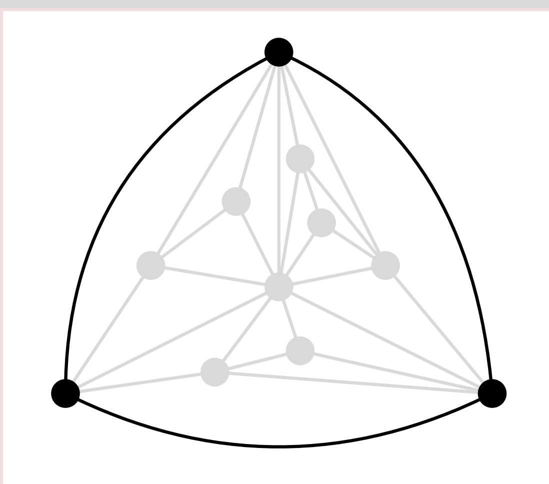

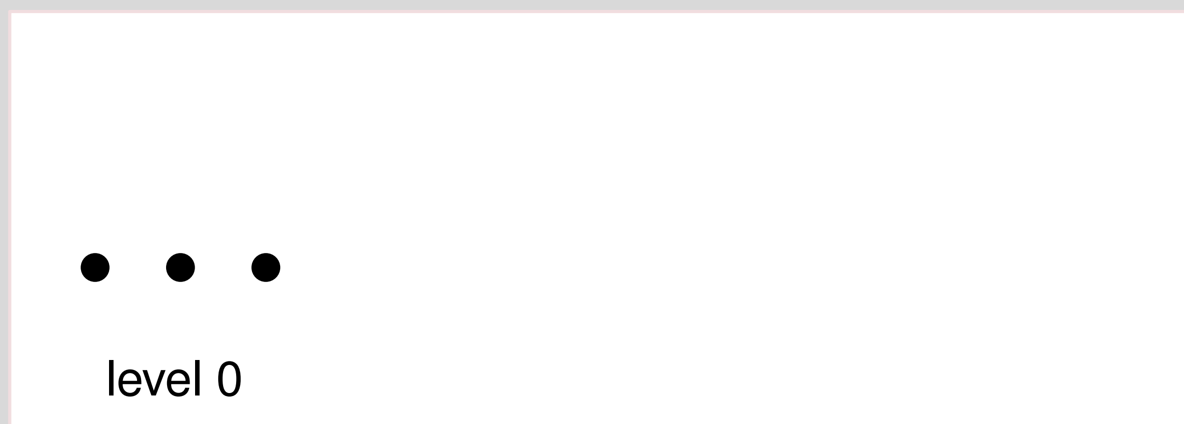

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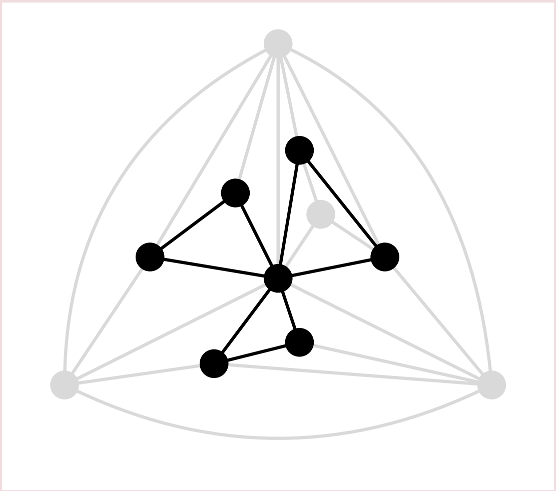
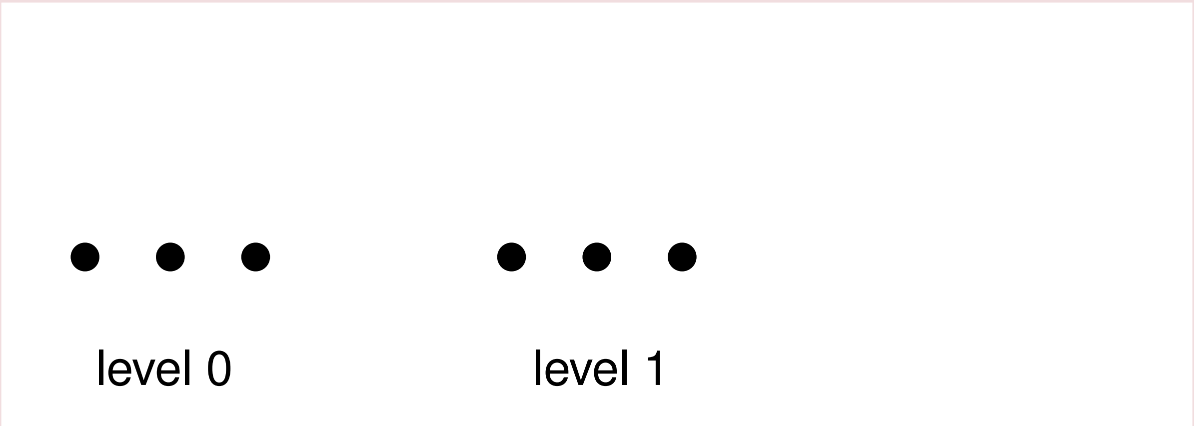
- level 1 is outerplanar





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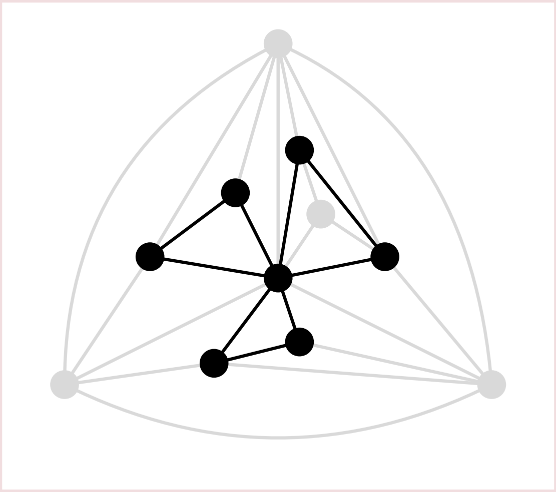

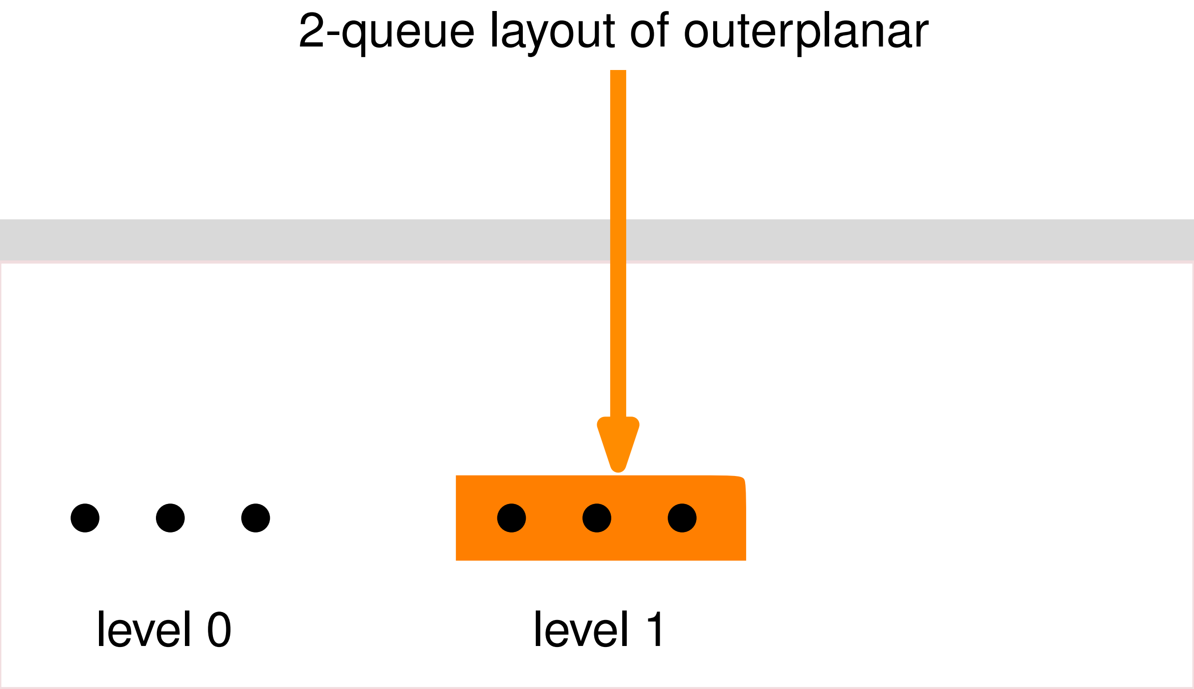

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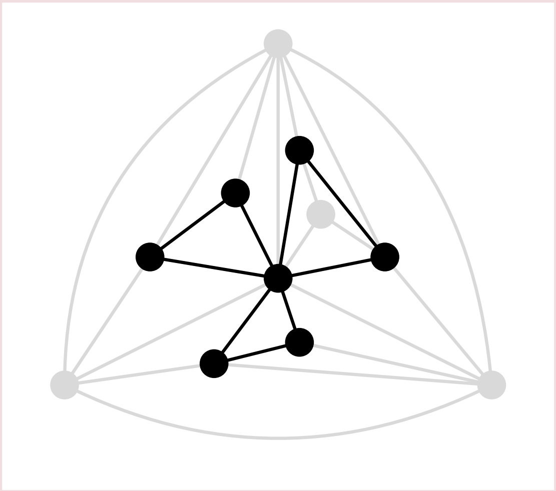
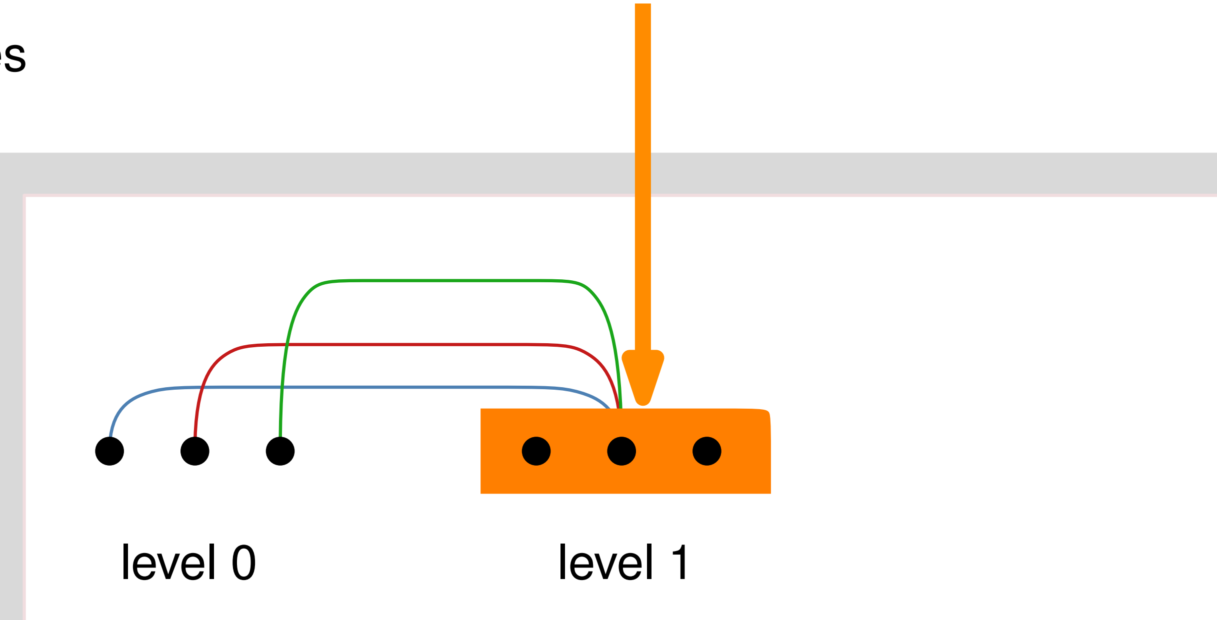
- leveling of vertices
- level-0 graph is triangle
- level 1 is outerplanar
- 3 queues for binding edges

2-queue layout of outerplanar





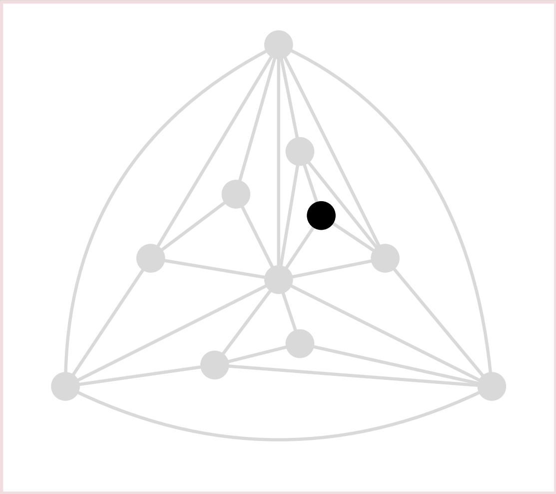
[1] Alam, Bekos, Gronemann, Kaufmann, Pupyrev:
Queue layouts of planar 3-trees (2020)

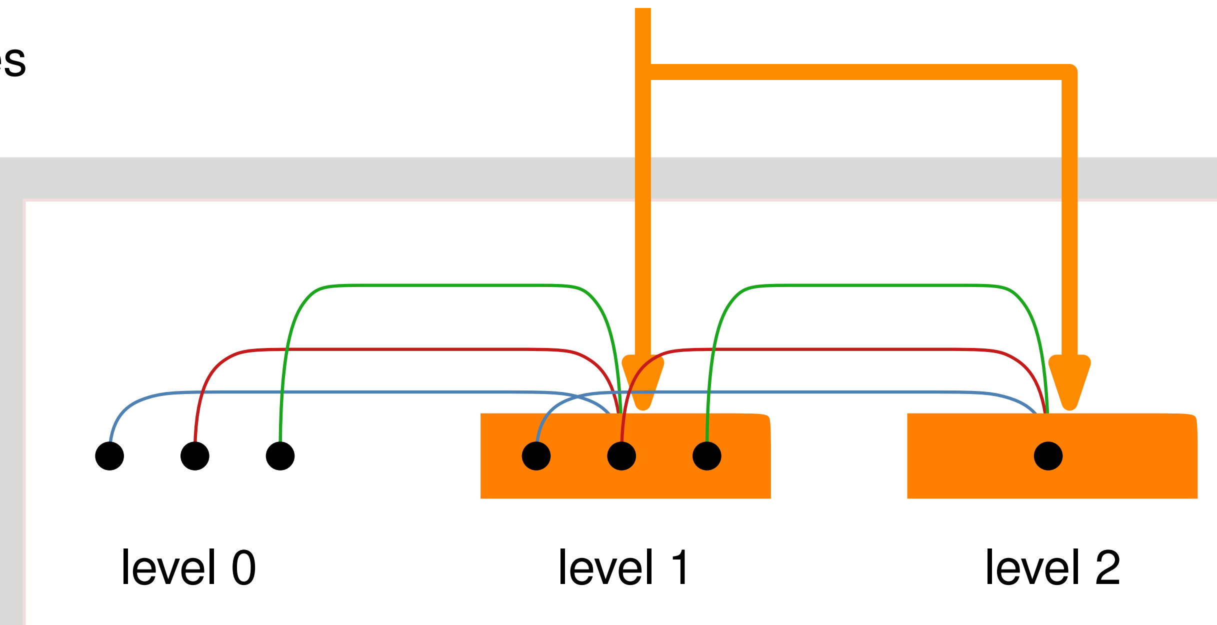
Step 2: 5-queue layout of H

Theorem[1]:
A planar 3-tree H has $\mathrm{qn}(H) \leq 5$

- maximal planar 3-tree
- leveling of vertices
- level-0 graph is triangle
- level 1 is outerplanar
- 3 queues for binding edges
- recurse in faces

2-queue layout of outerplanar





[1] Alam, Bekos, Gronemann, Kaufmann, Pupyrev: *Queue layouts of planar 3-trees* (2020)

Step 2: 2-queue layout of outerplanar

Theorem[1]:

Outerplanar graph G has $\mathrm{qn}(G) = 2$

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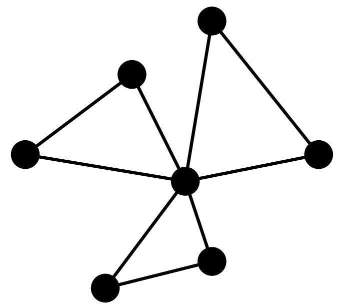

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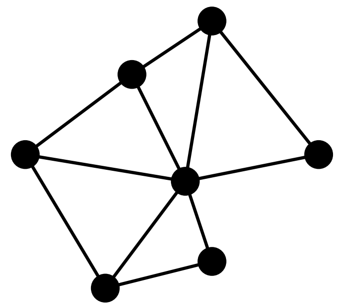

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Step 2: 2-queue layout of outerplanar

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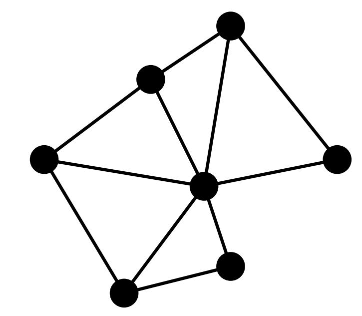

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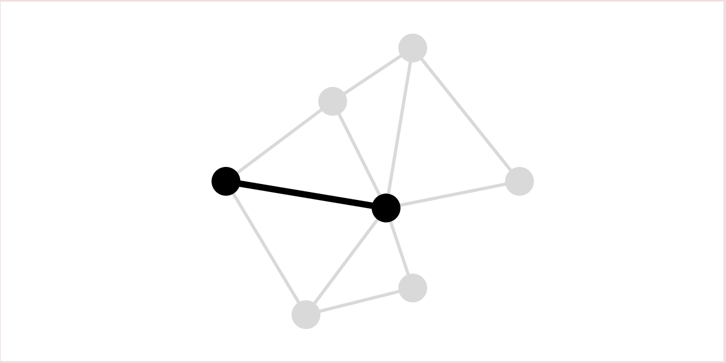

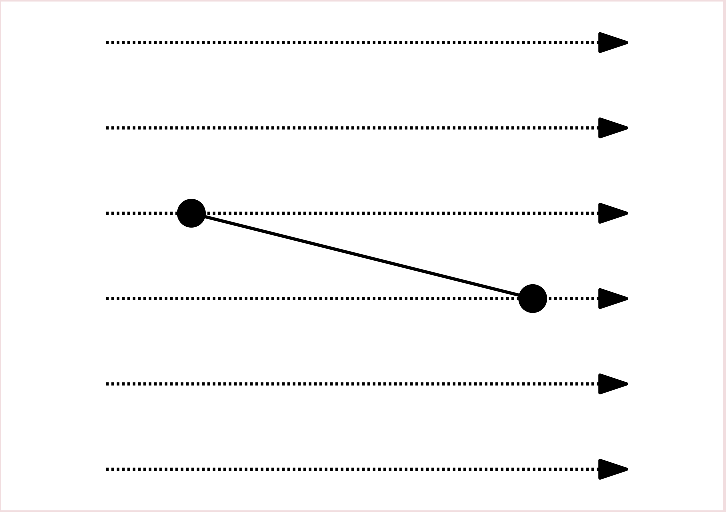

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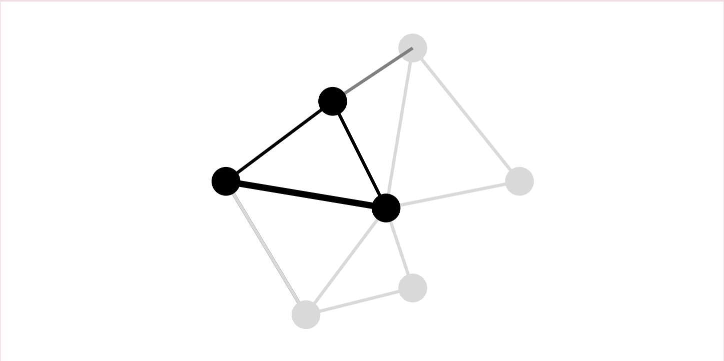

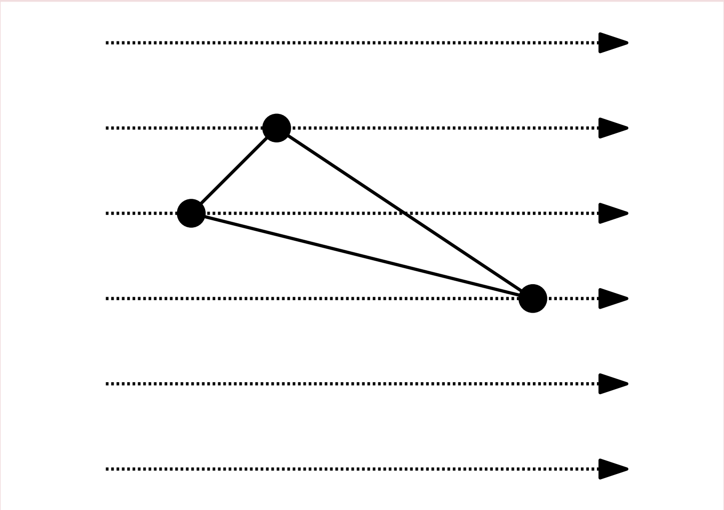

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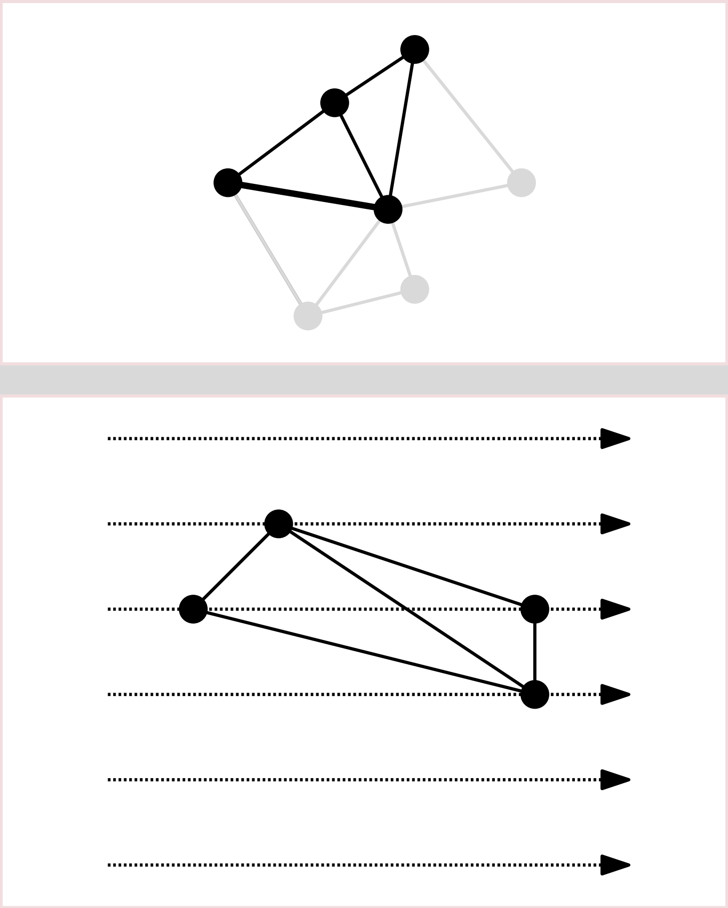

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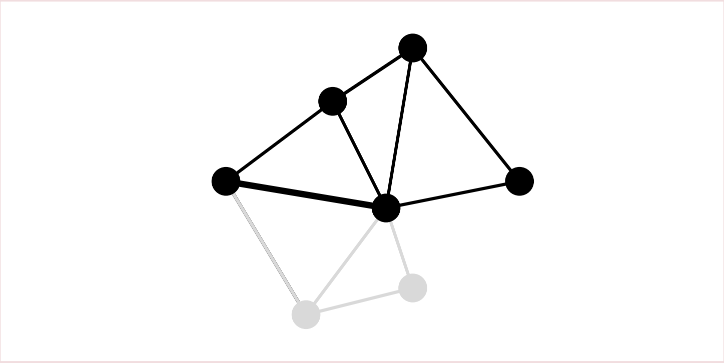

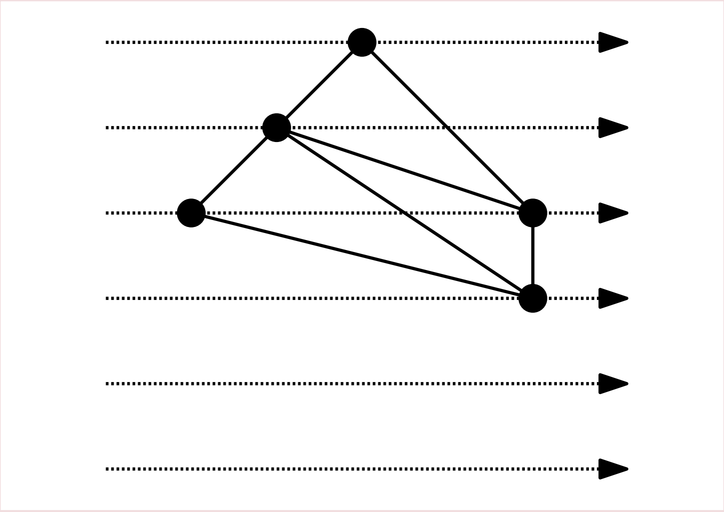

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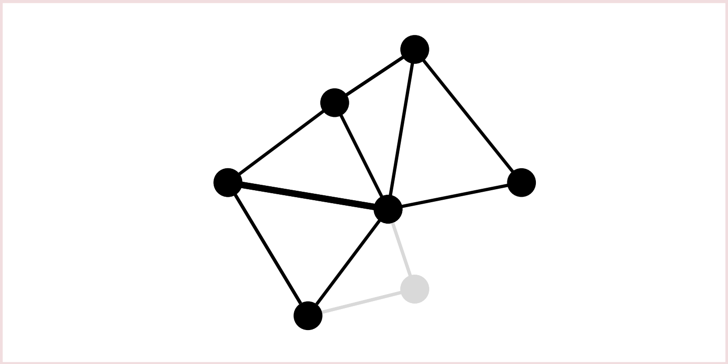

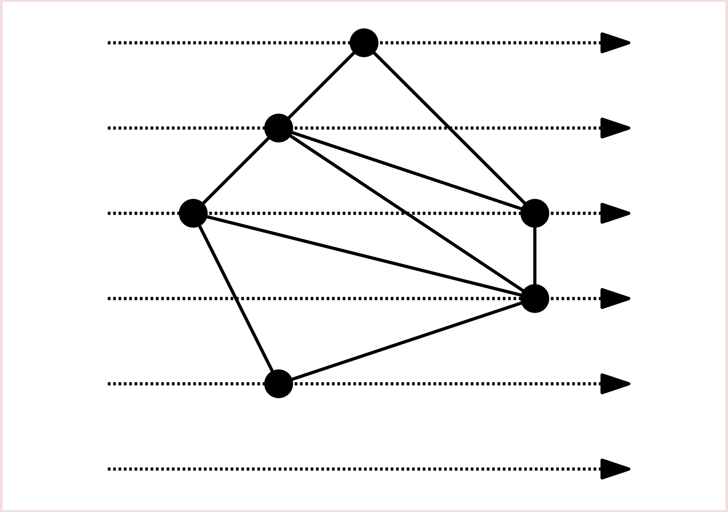

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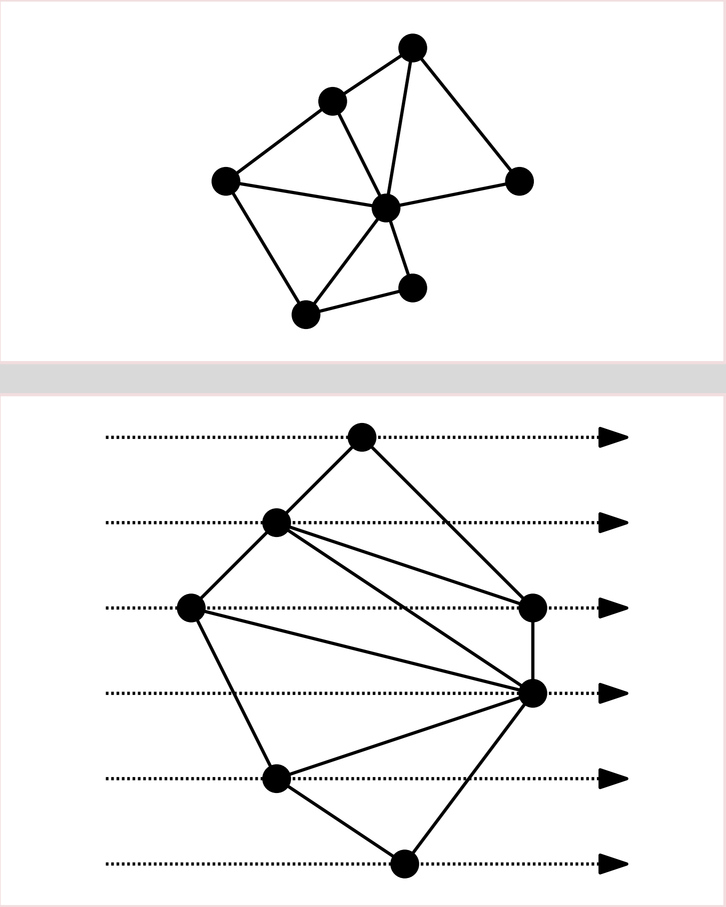

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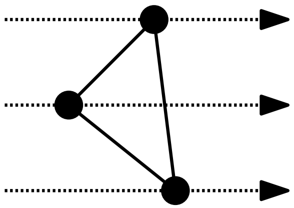
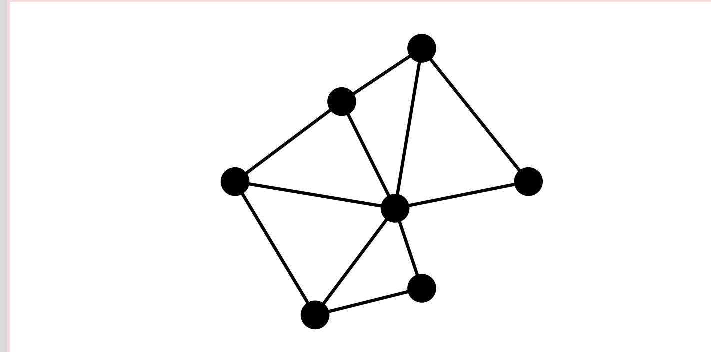
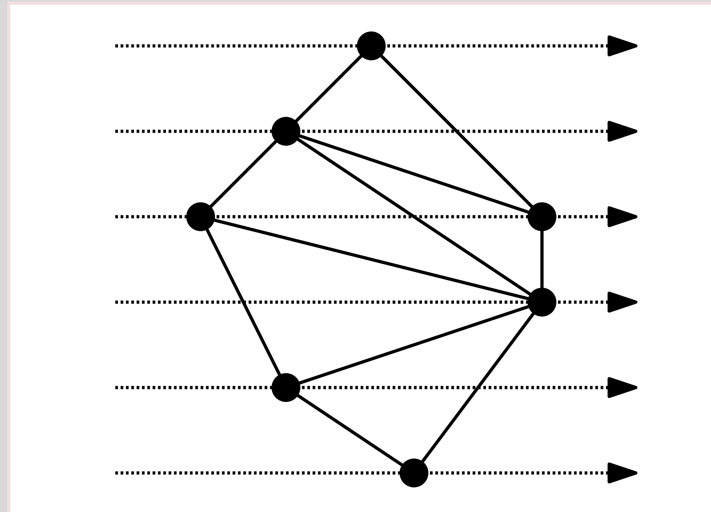
- edges have span 1 or 2
- face has a top, middle and bottom vertex







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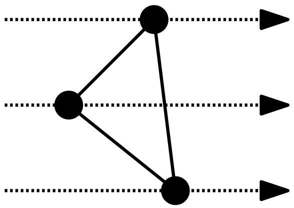

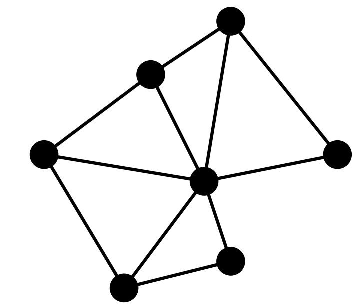

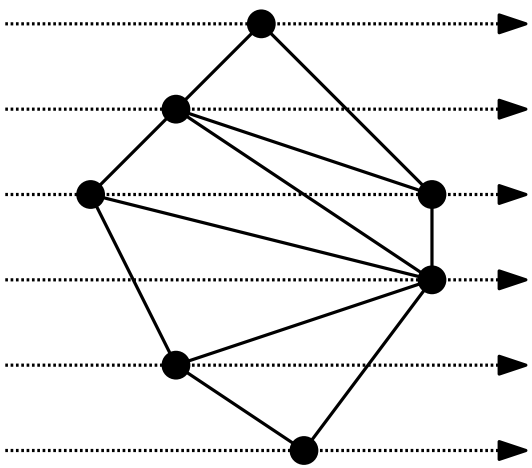

2-queue layout

- span 1 edges on one queue
- span 2 edges on one queue

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Step 2: 2-queue layout of outerplanar

Theorem[1]:

Outerplanar graph *G* has qn(*G*) = 2

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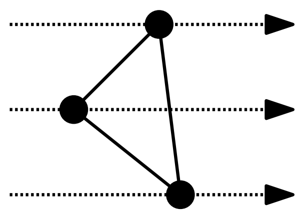
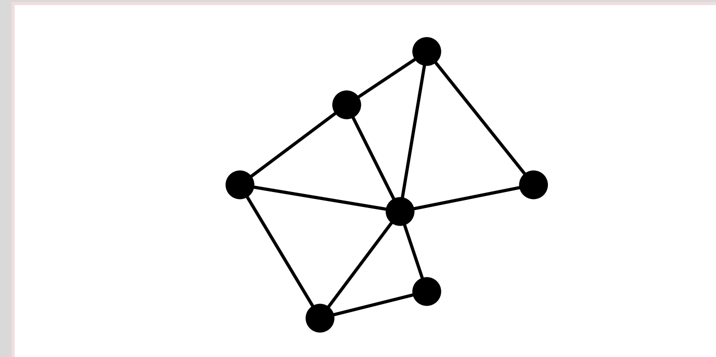
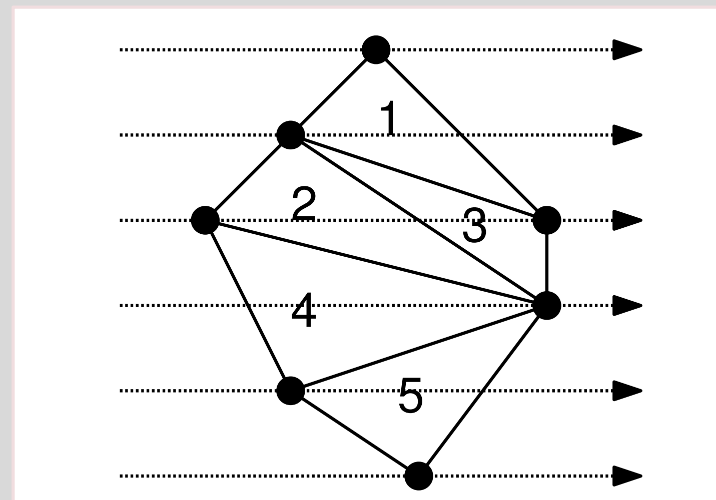


2-queue layout

- span 1 edges on one queue
- span 2 edges on one queue
- faces are ordered





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Overview

Theorem [Dujmović et al.]:

$$qn(G) \leq 3 \cdot 3 \cdot qn(H) + 4 \leq 49$$

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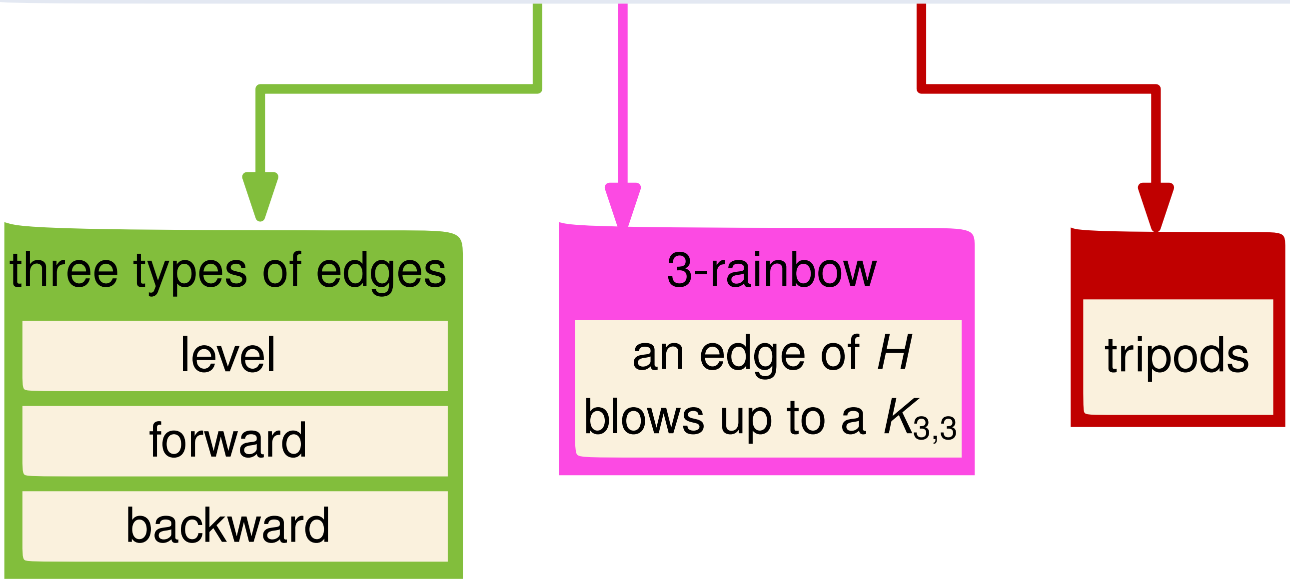

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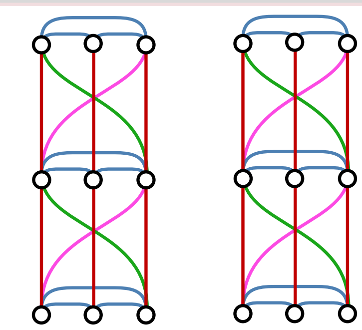
tripods

- 5-queue layout of H determines the order of the tripods



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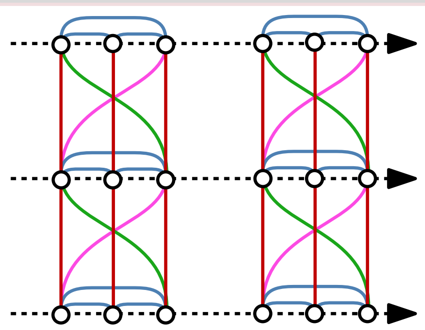

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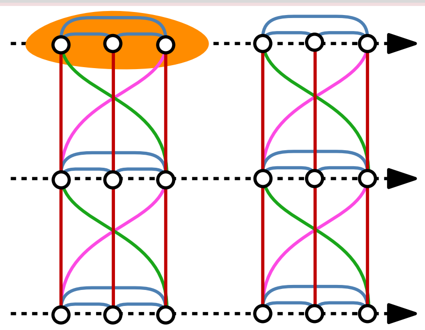
tripods

- 5-queue layout of H determines the order of the tripods
- BFS-leveling of G is a partial order
- each 3 vertices of the same tripod and on same BFS-level are unordered



Improvement

Main Idea:

order the paths of each tripod

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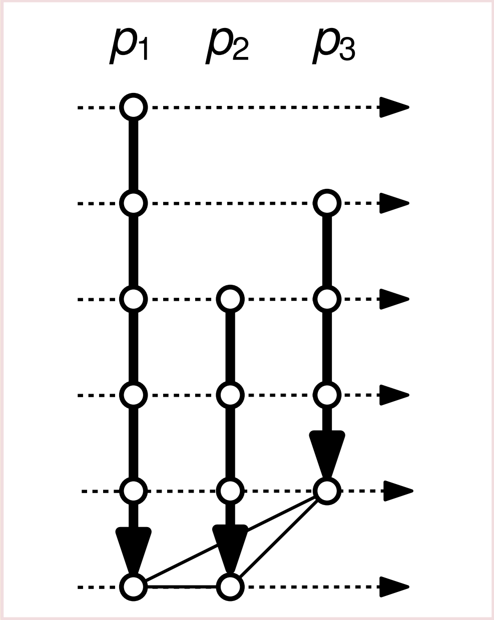

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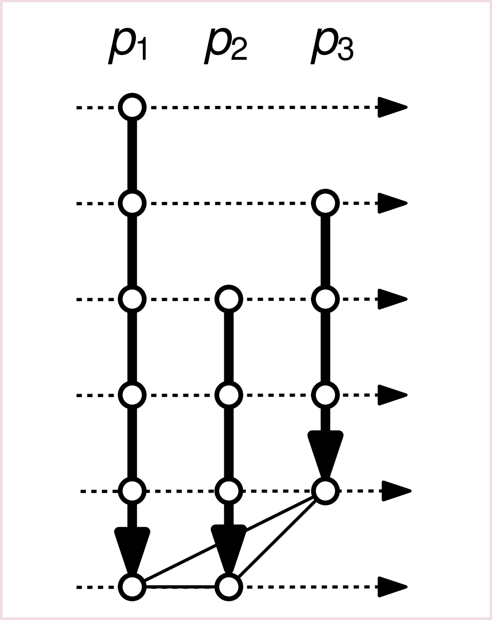

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  (level edges of $P_k \boxtimes K_3$)

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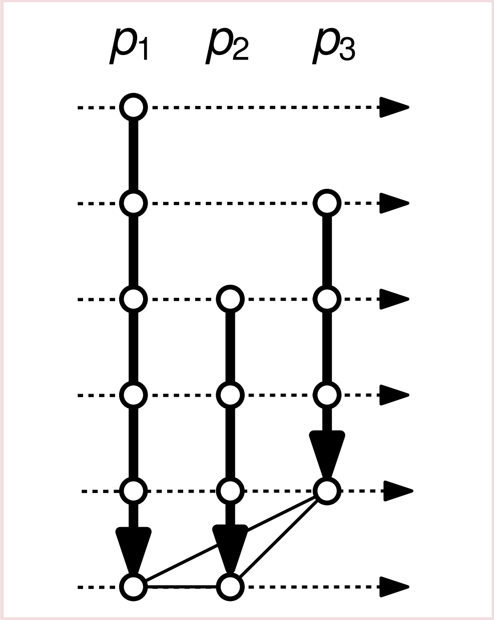

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Claim

- binding (and vertex) edges
  create no 3-rainbow

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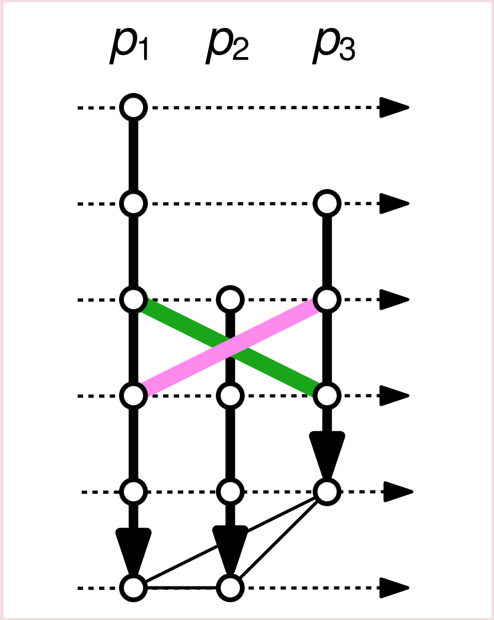

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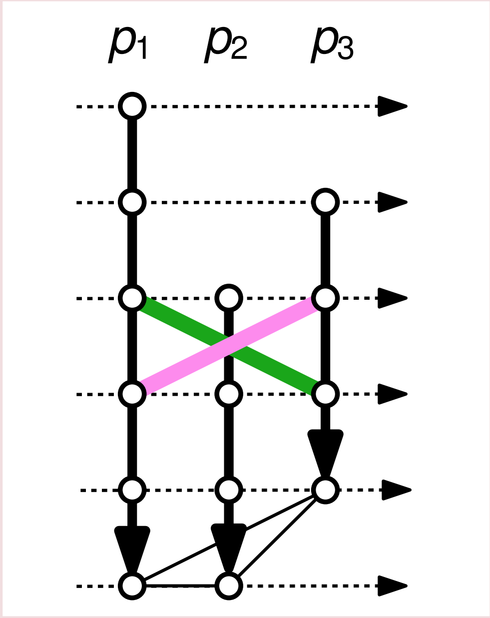

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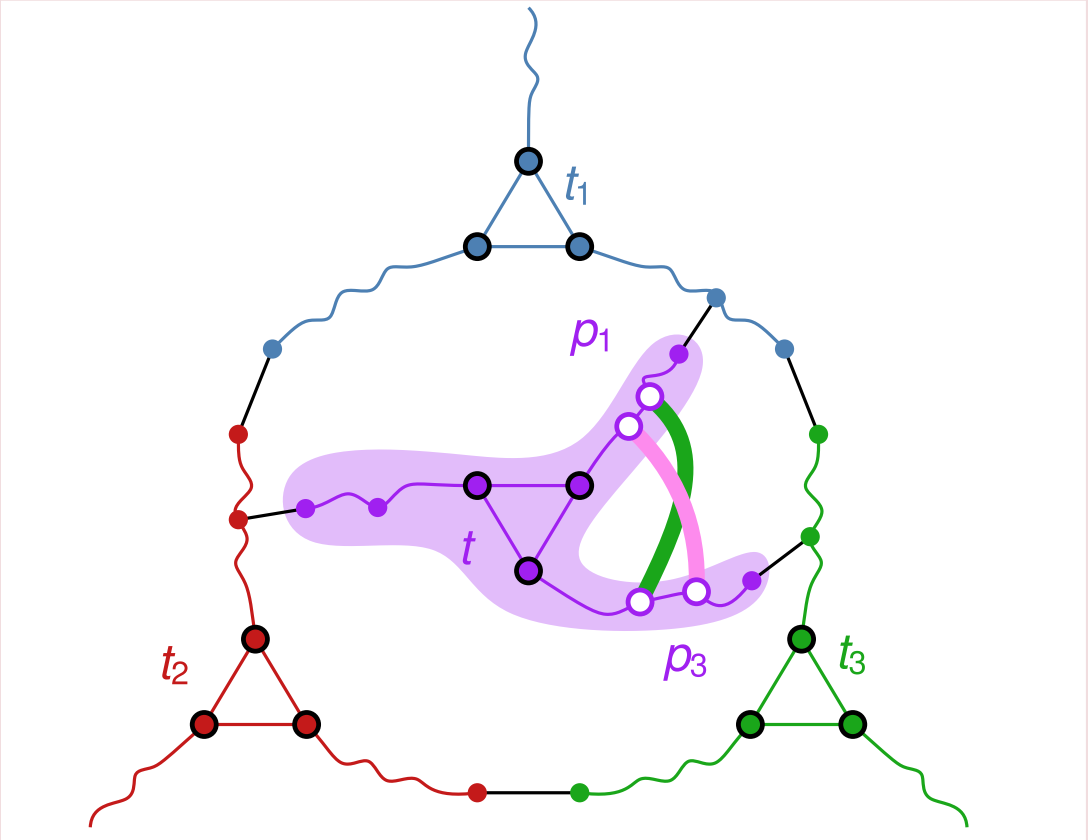

otherwise planarity is violated

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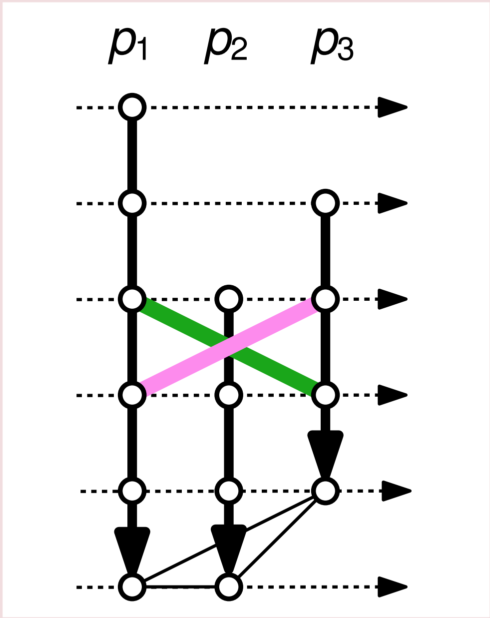

- level edges on one queue (level edges of $P_k \boxtimes K_3$)

Tripods need 3 queues upper bound reduced to 48

Claim

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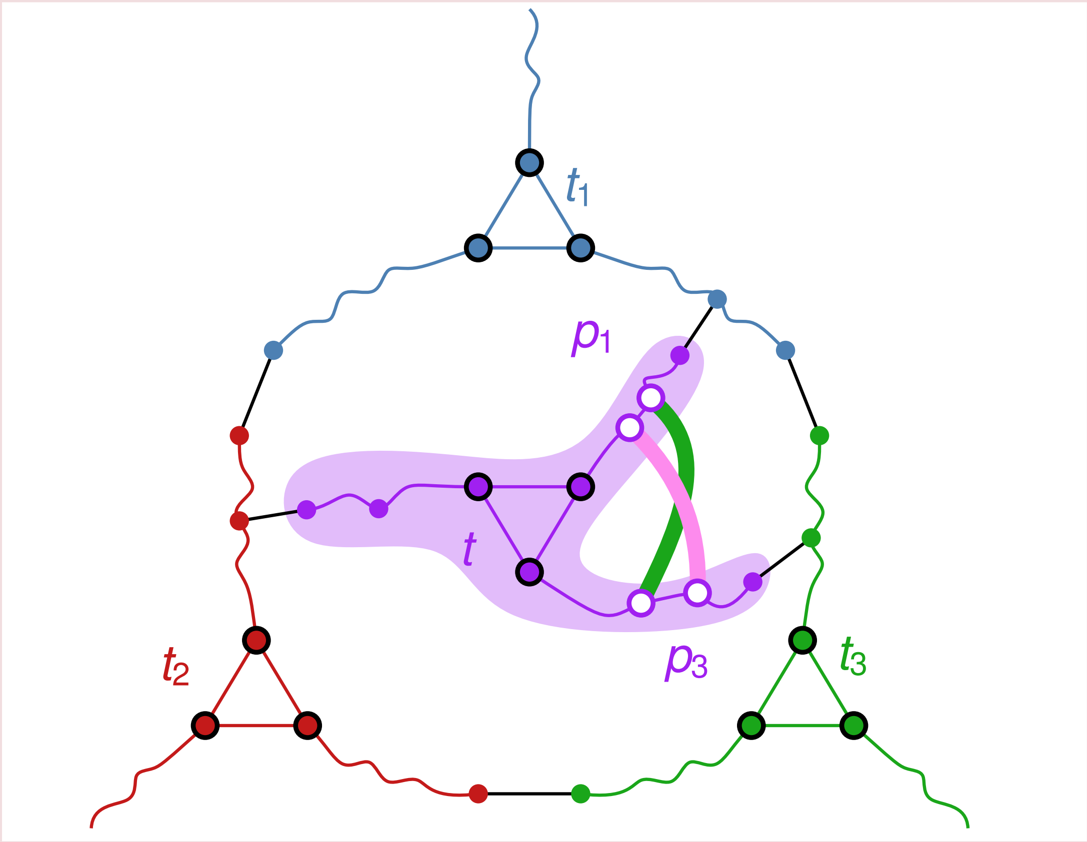

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Goal:

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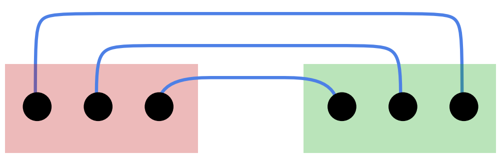

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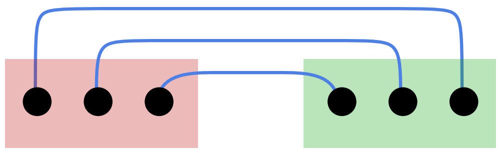
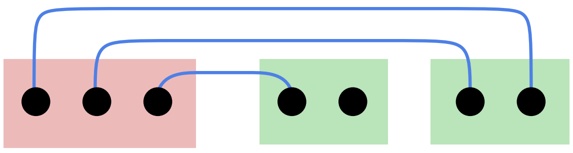


two tripods



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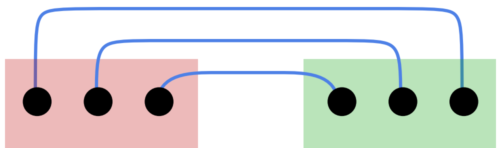

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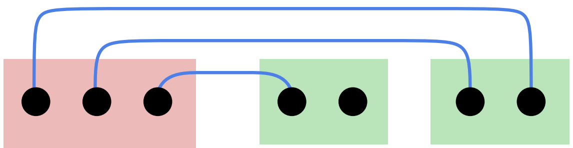
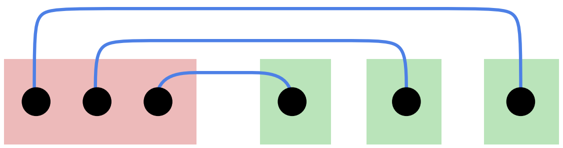


three tripod



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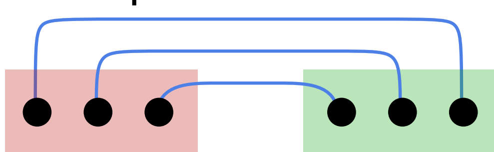
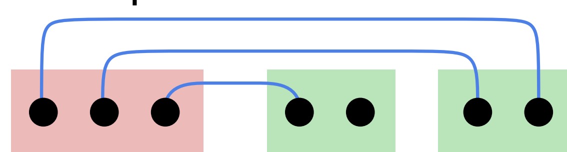
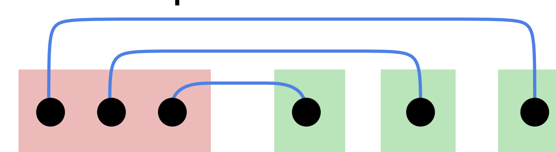
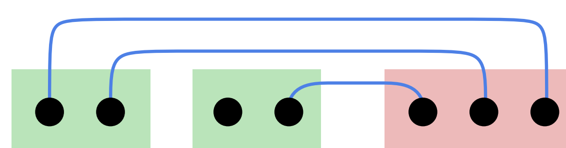


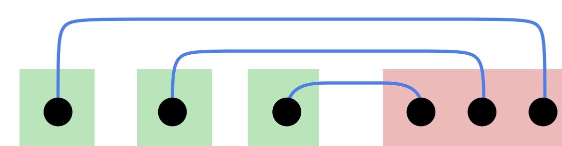
two tripods



three tripod







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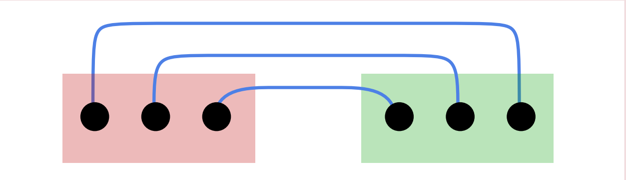
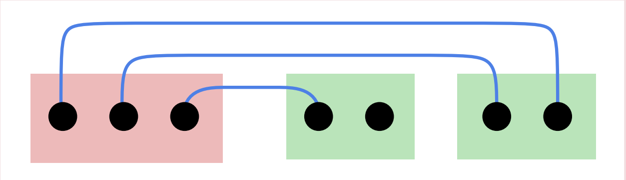
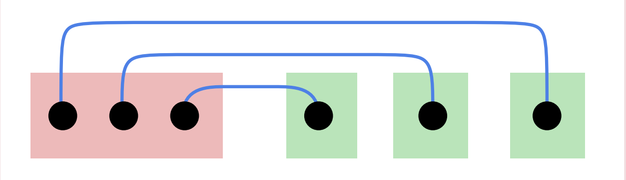
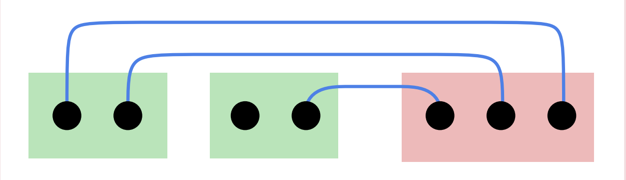
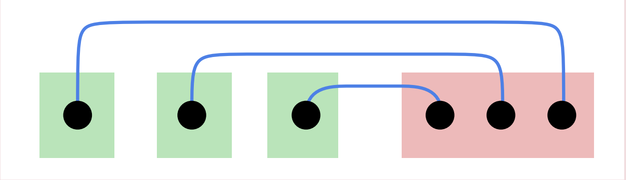
Types of 3-rainbows





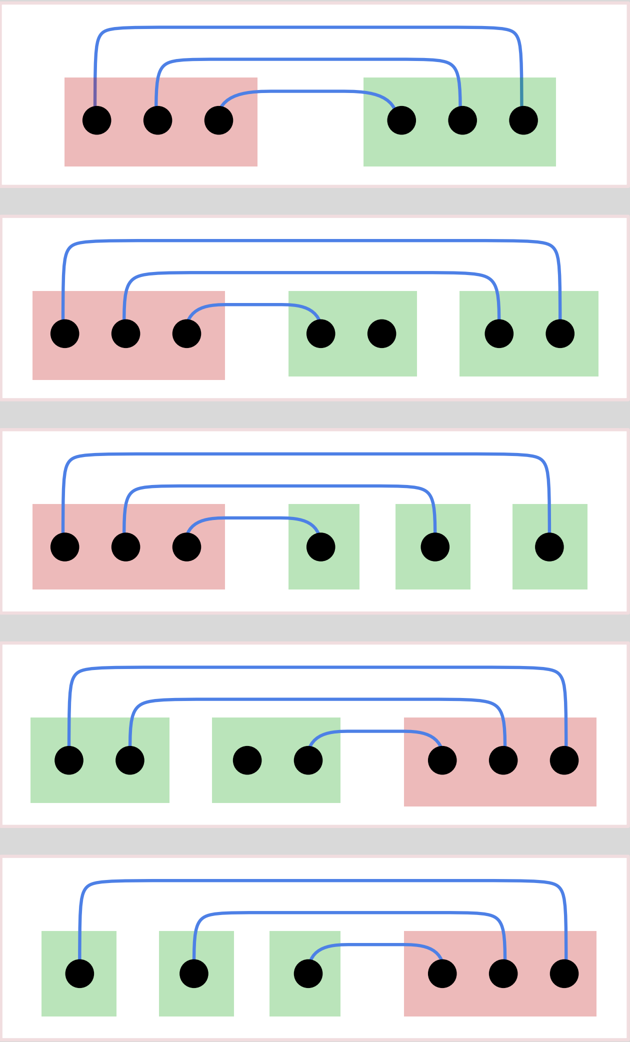






Improvement

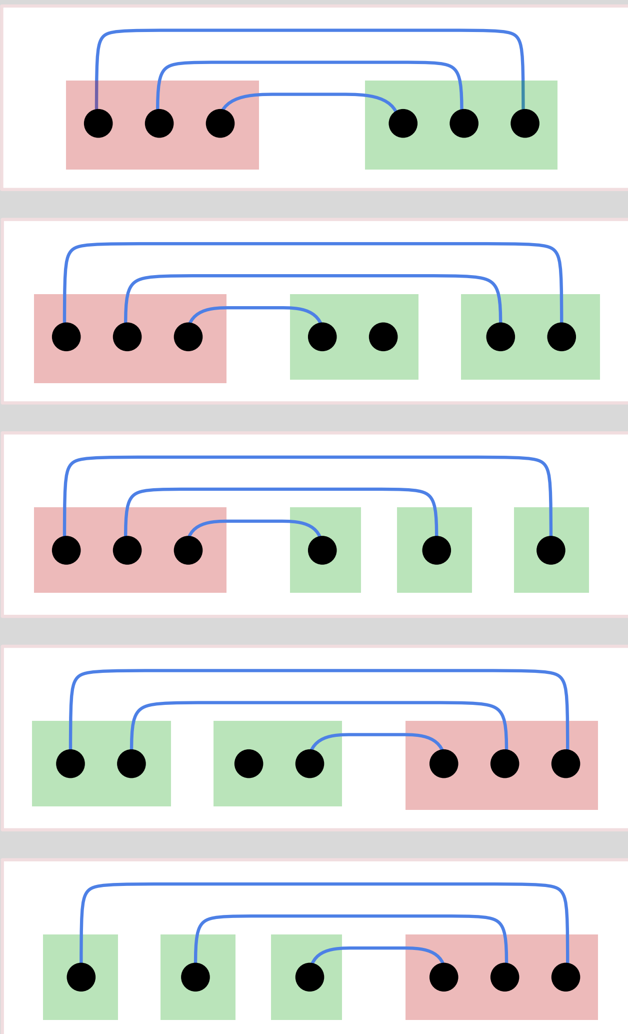
Types of 3-rainbows



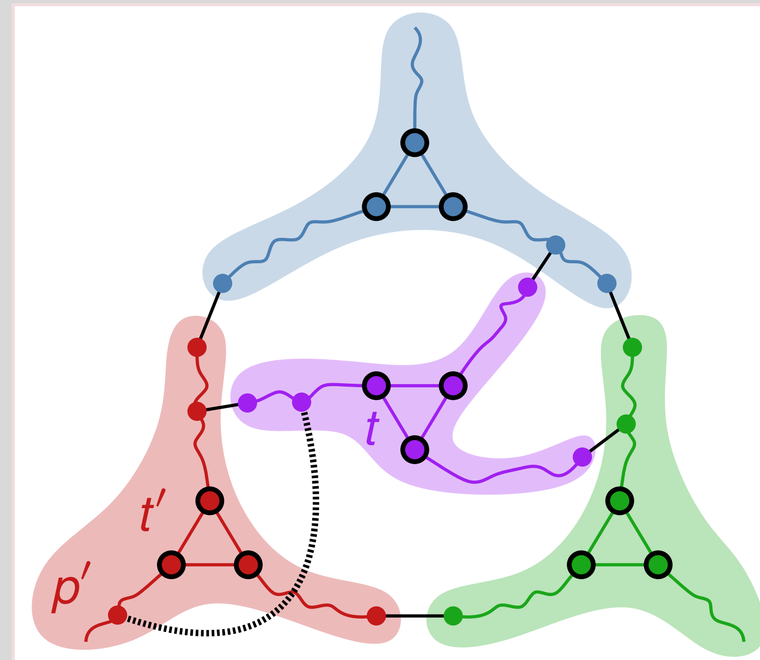
- for two adjacent tripods one is parent of the other

Improvement

Types of 3-rainbows



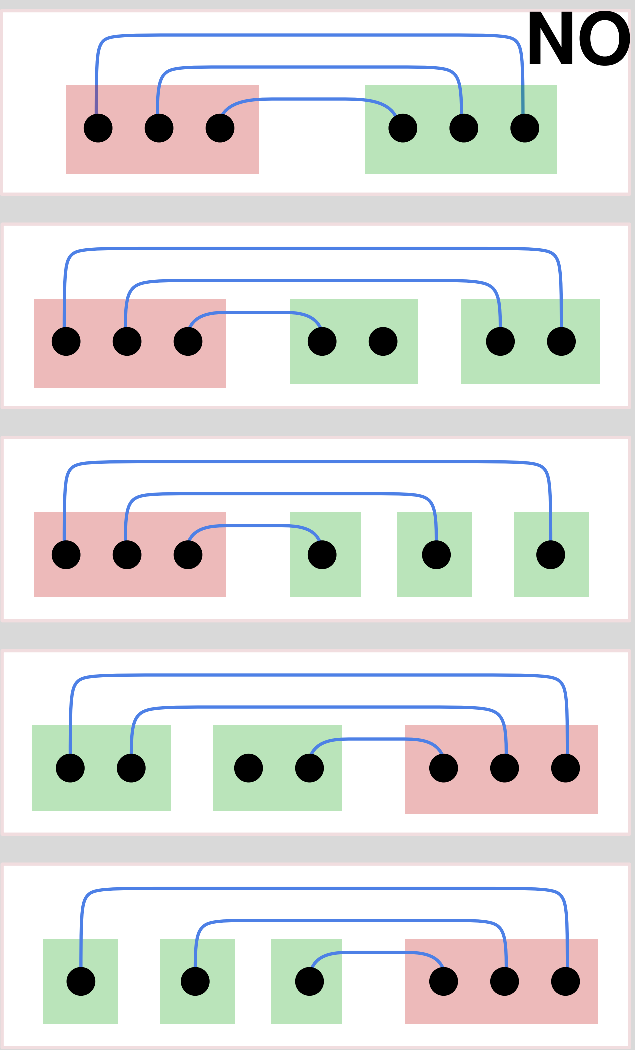
- for two adjacent tripods one is parent of the other
- if t' parent of t, t is not adjacent to a path p' of t'



otherwise planarity is violated

Improvement

Types of 3-rainbows



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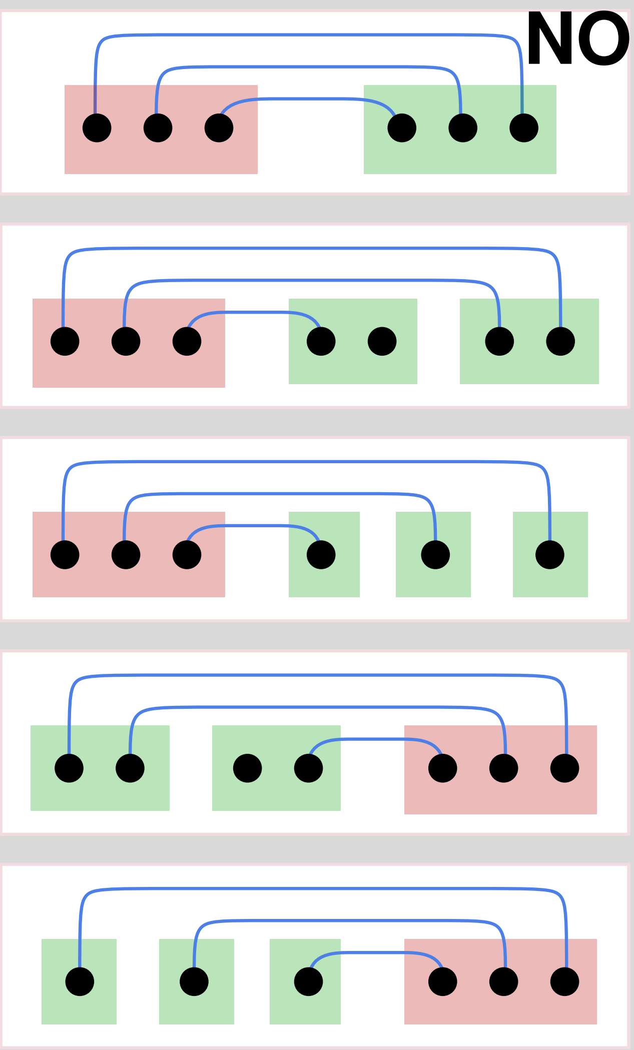

- for two adjacent tripods one is parent of the other
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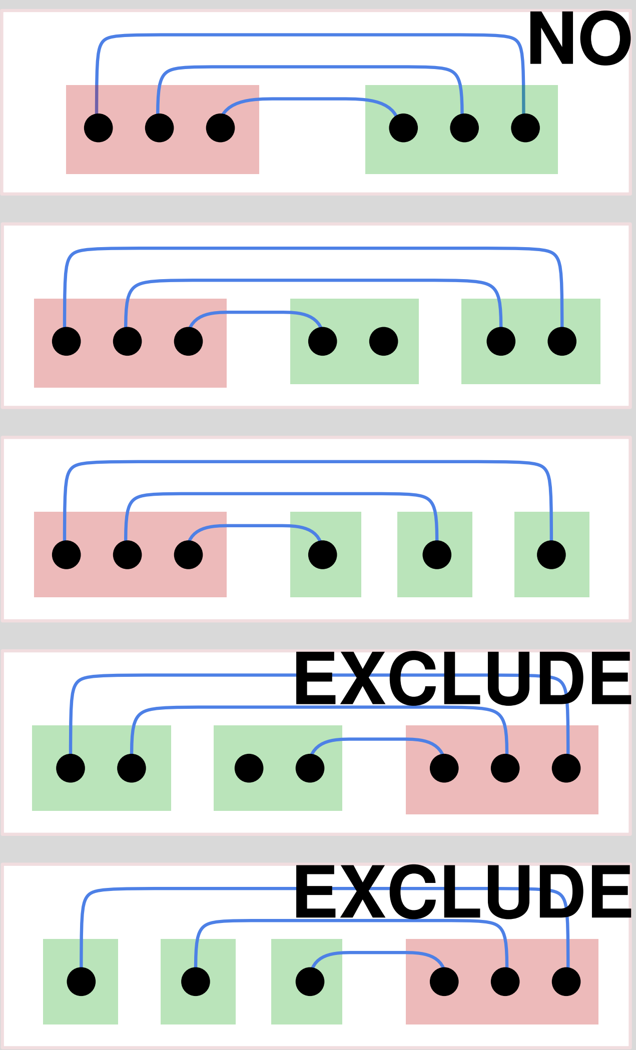
- left and right might conflict

Improvement

Types of 3-rainbows



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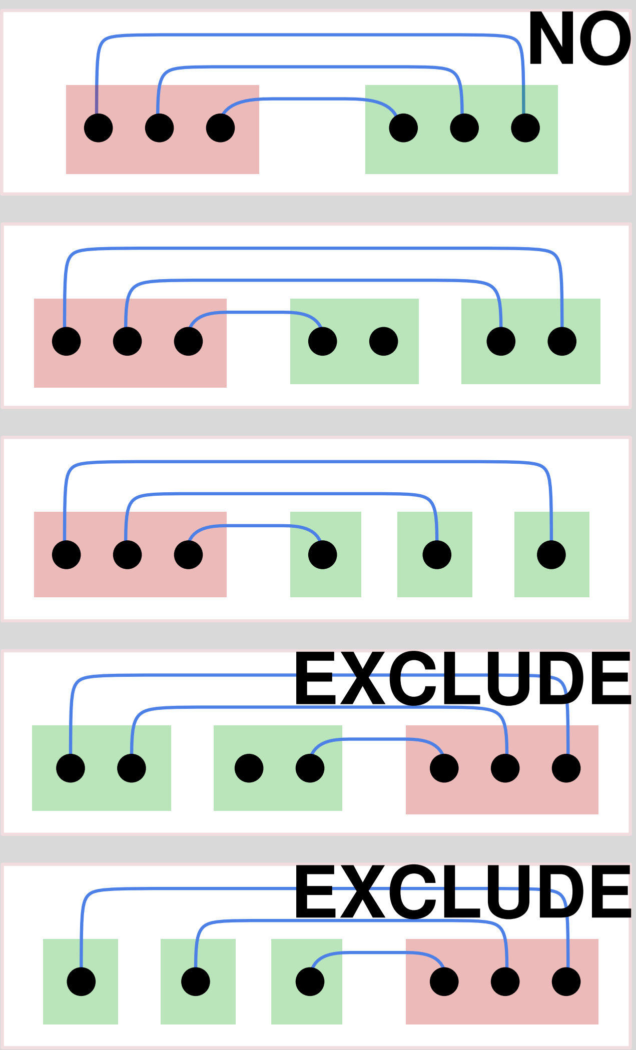
$\rightarrow$ consider only following tripods

Improvement

Types of 3-rainbows



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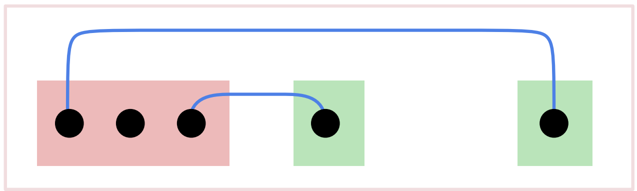
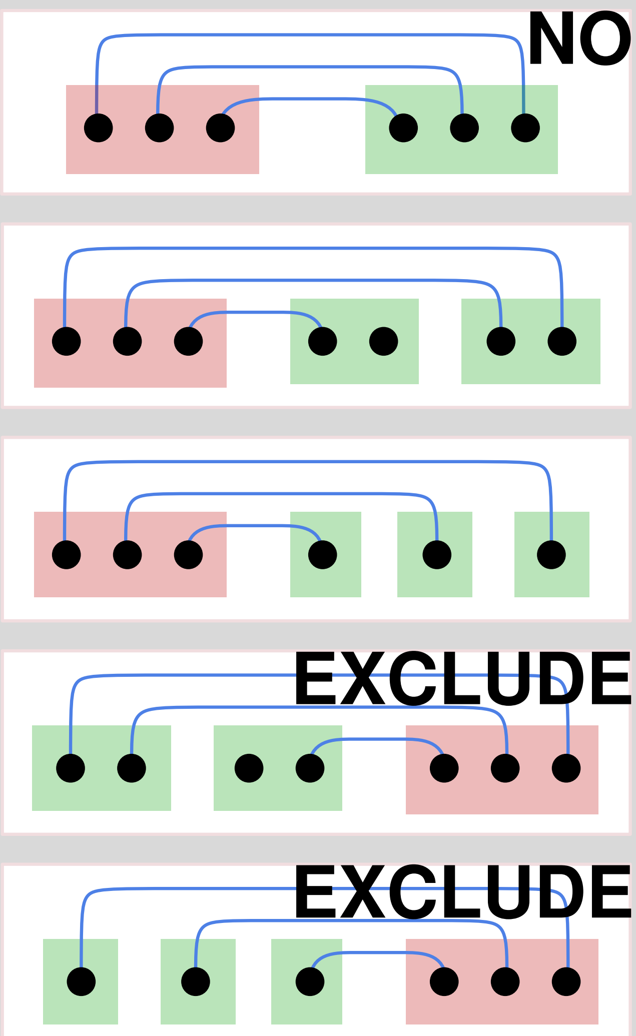


- avoid one of the two edges

Improvement

Types of 3-rainbows



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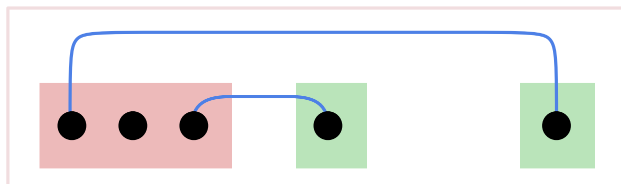

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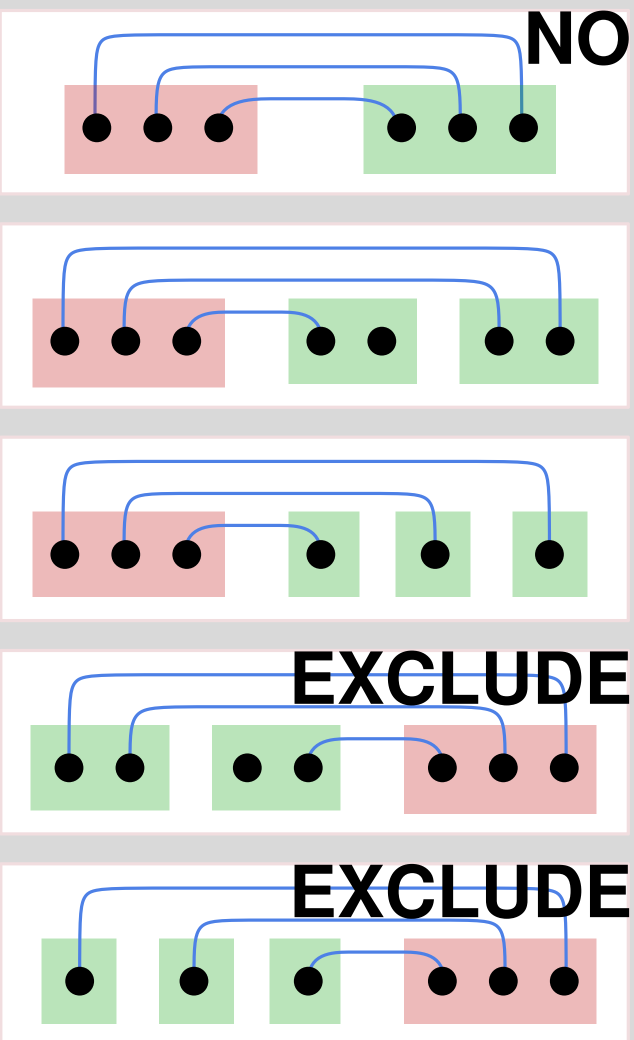
$\rightarrow$ consider only children

Improvement

Types of 3-rainbows



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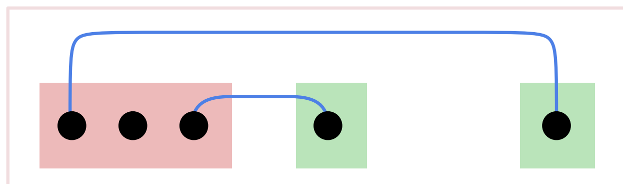


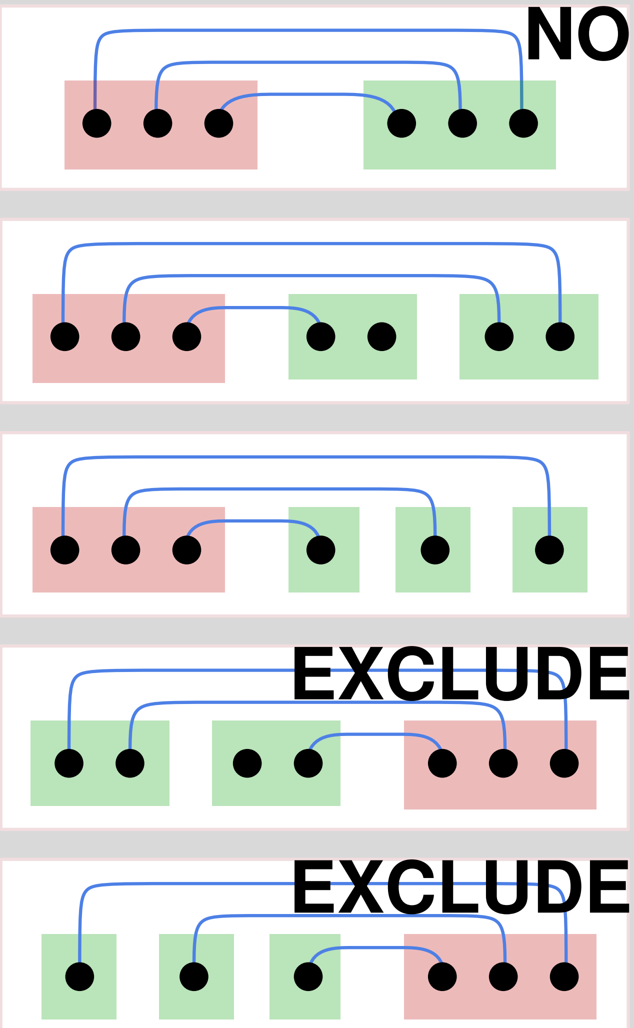
- avoid one of the two edges

$\rightarrow$ consider only children

- children are grouped based on which paths they don't see
- following tripods form a series of groups

Improvement

Types of 3-rainbows



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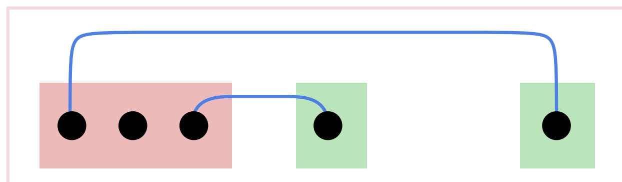

- avoid one of the two edges

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- children are grouped based on which paths they don't see
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$\rightarrow$ require two groups

Restrictions

Restrictions

1. Consider only following tripods
2. Split considered nodes into two groups
3. Consider only children

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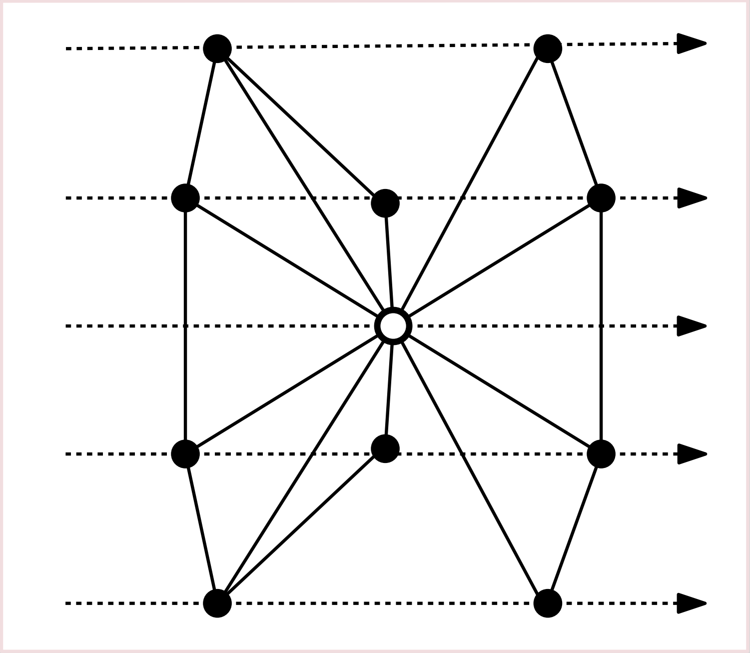

Restrictions

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1. Consider only following tripods
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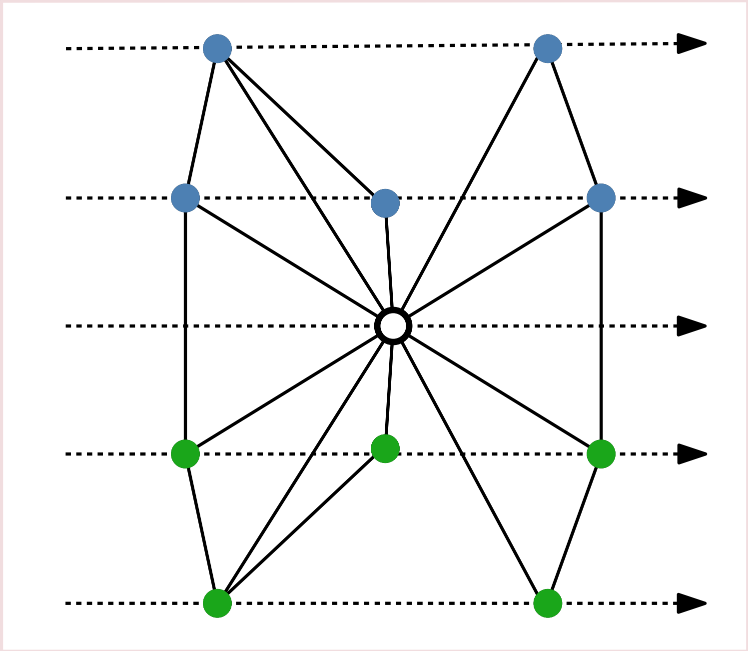
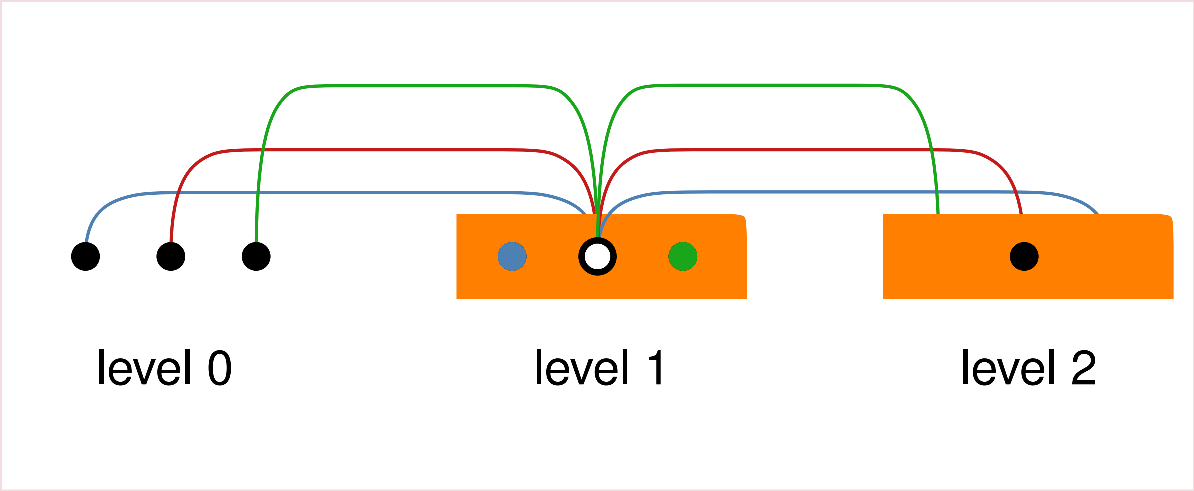
- blue vertices precede and green follow
- exclude queues of the outerplanar subgraph
- other queues satisfy 1.





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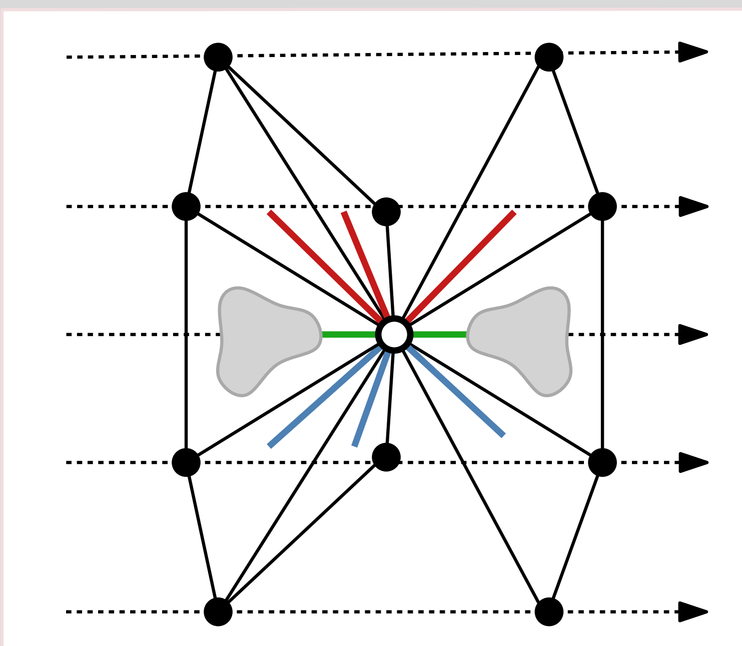
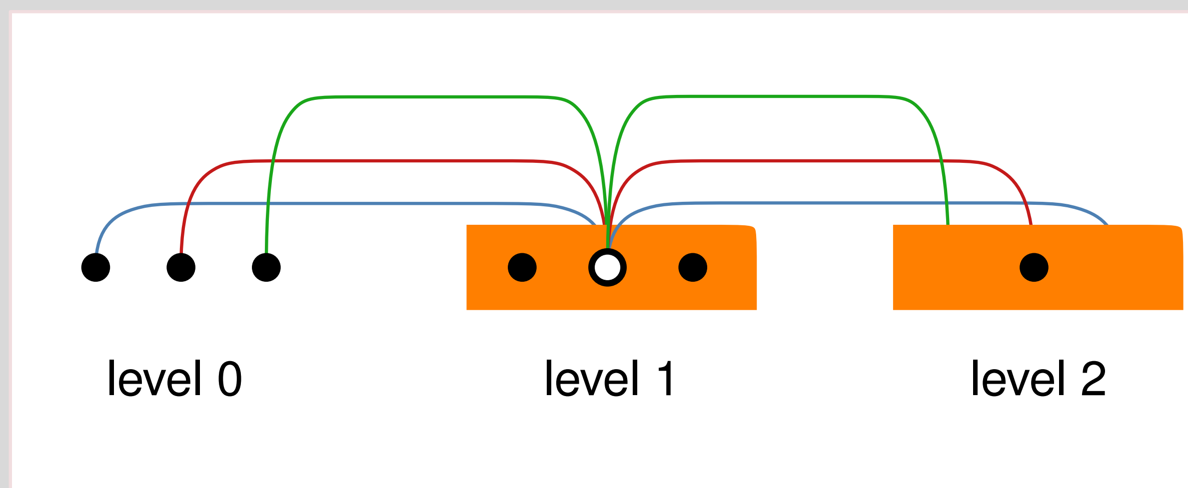
- green neighbors form two clusters





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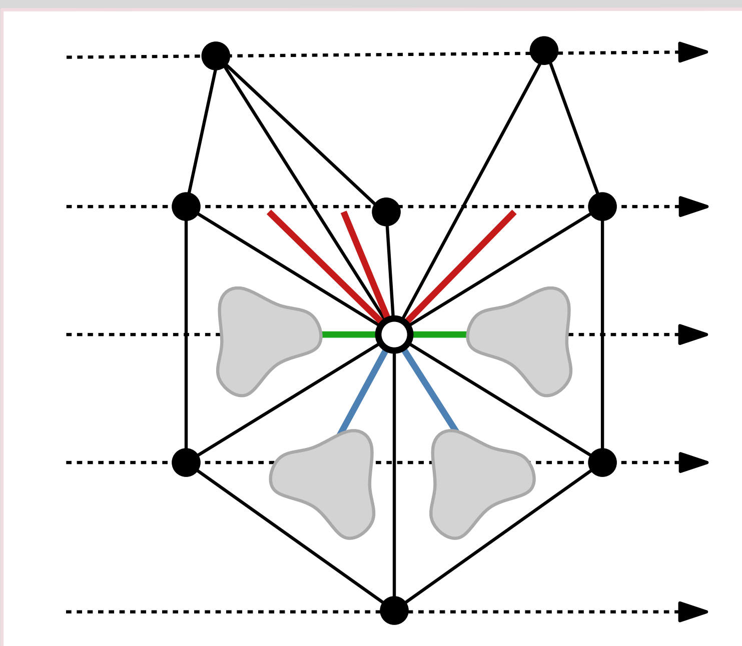
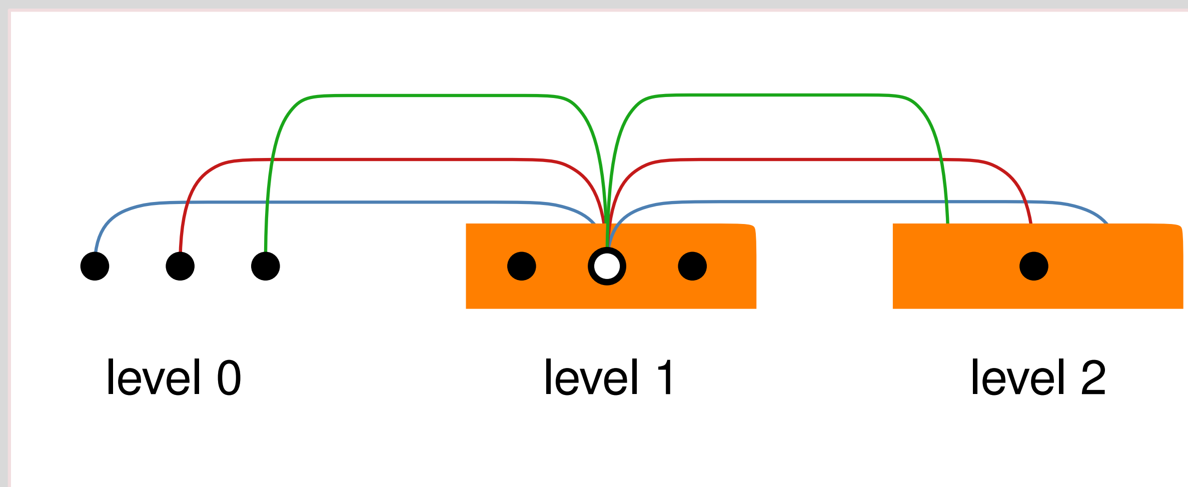
- green neighbors form two clusters
- do the same for the blue





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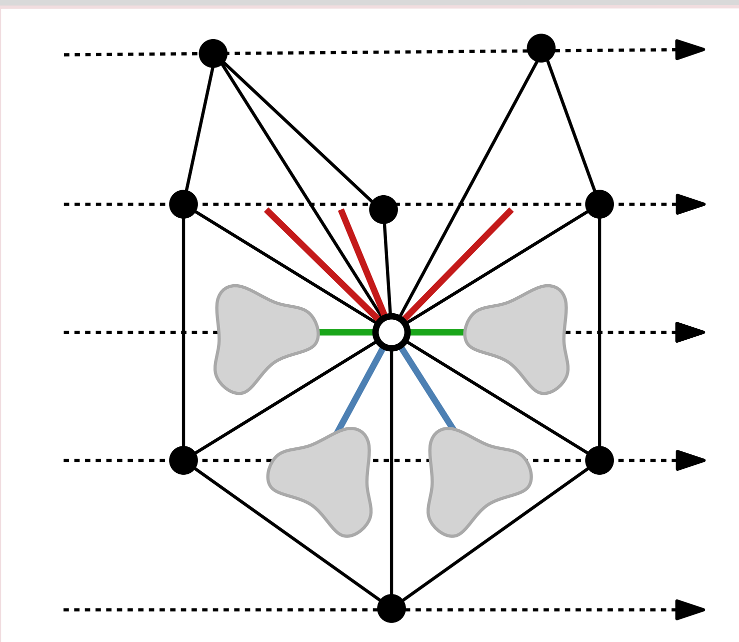
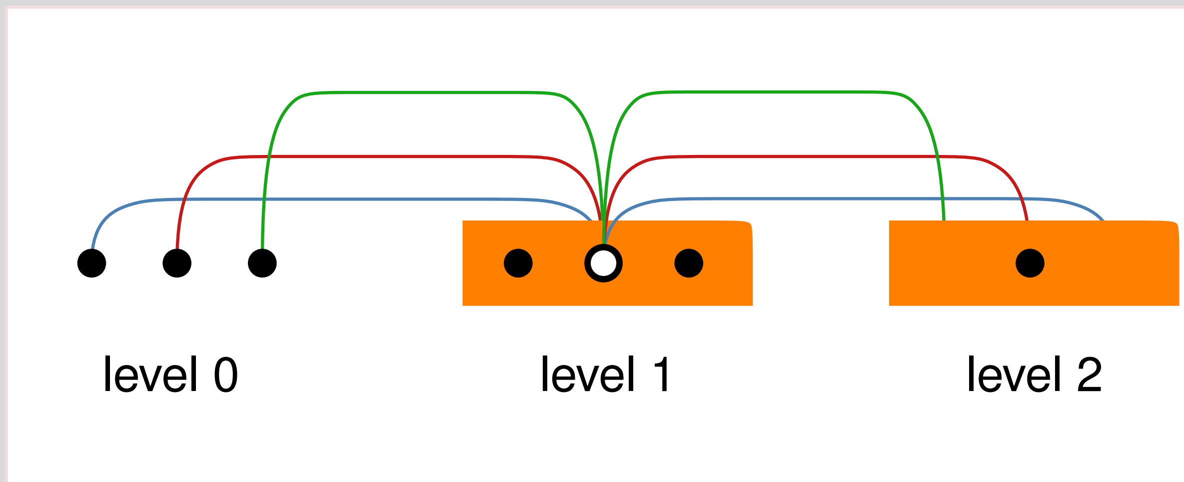


we can improve on two queues of the 5-queue layout of *H*



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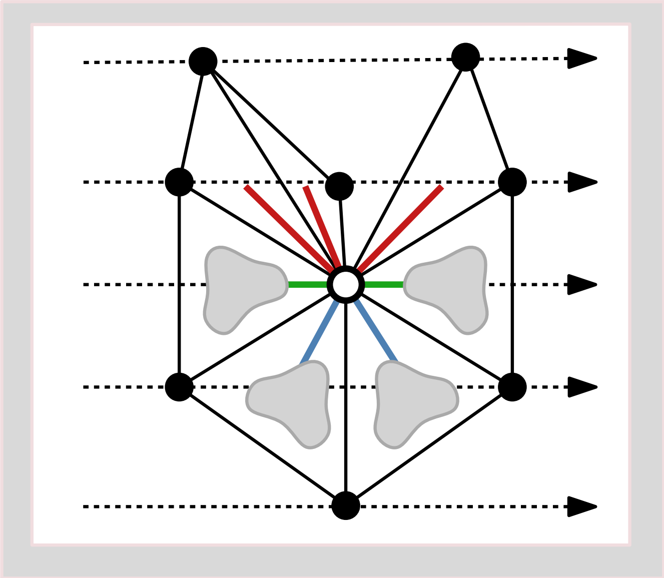

Restrictions

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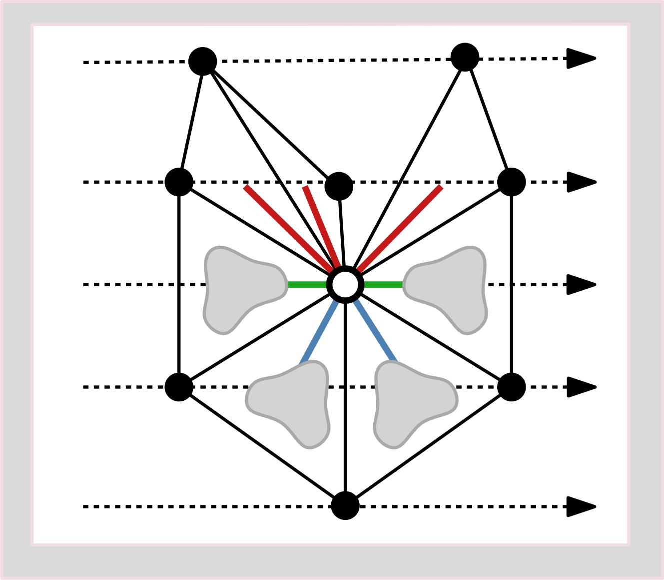
3. Consider only children

- green and blue clusters to be children
- each cluster to form a group,
  i.e. to not see a path of the tripod



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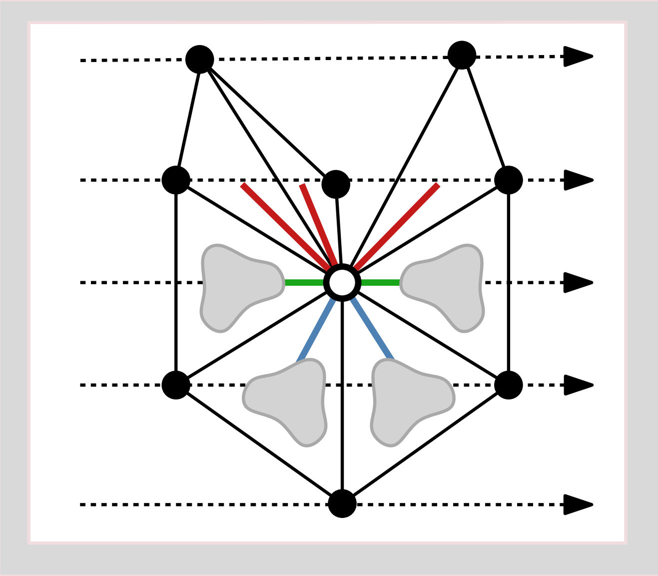

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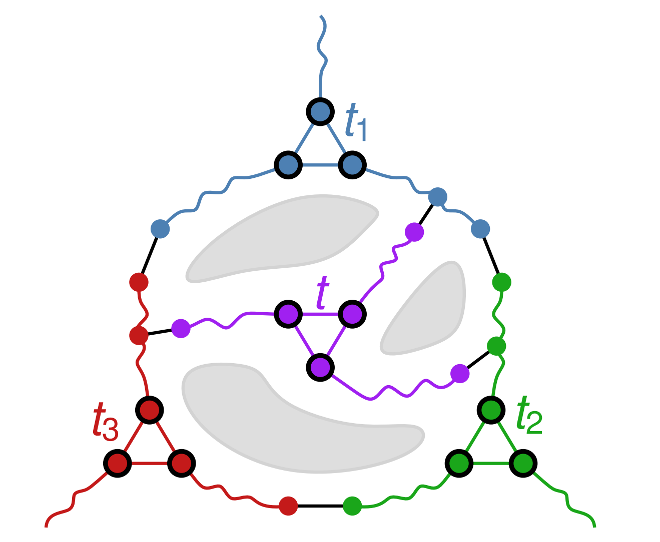

- each cluster to be in a shaded region
- graph *H* must reflect planarity of *G*
- graph *H* must respect parent-child relation
- the level of parent can't be greater than the child's

Improvement

- Contract tripods in *G*
  - preserve embedding
  - remove homotopic parallel edges

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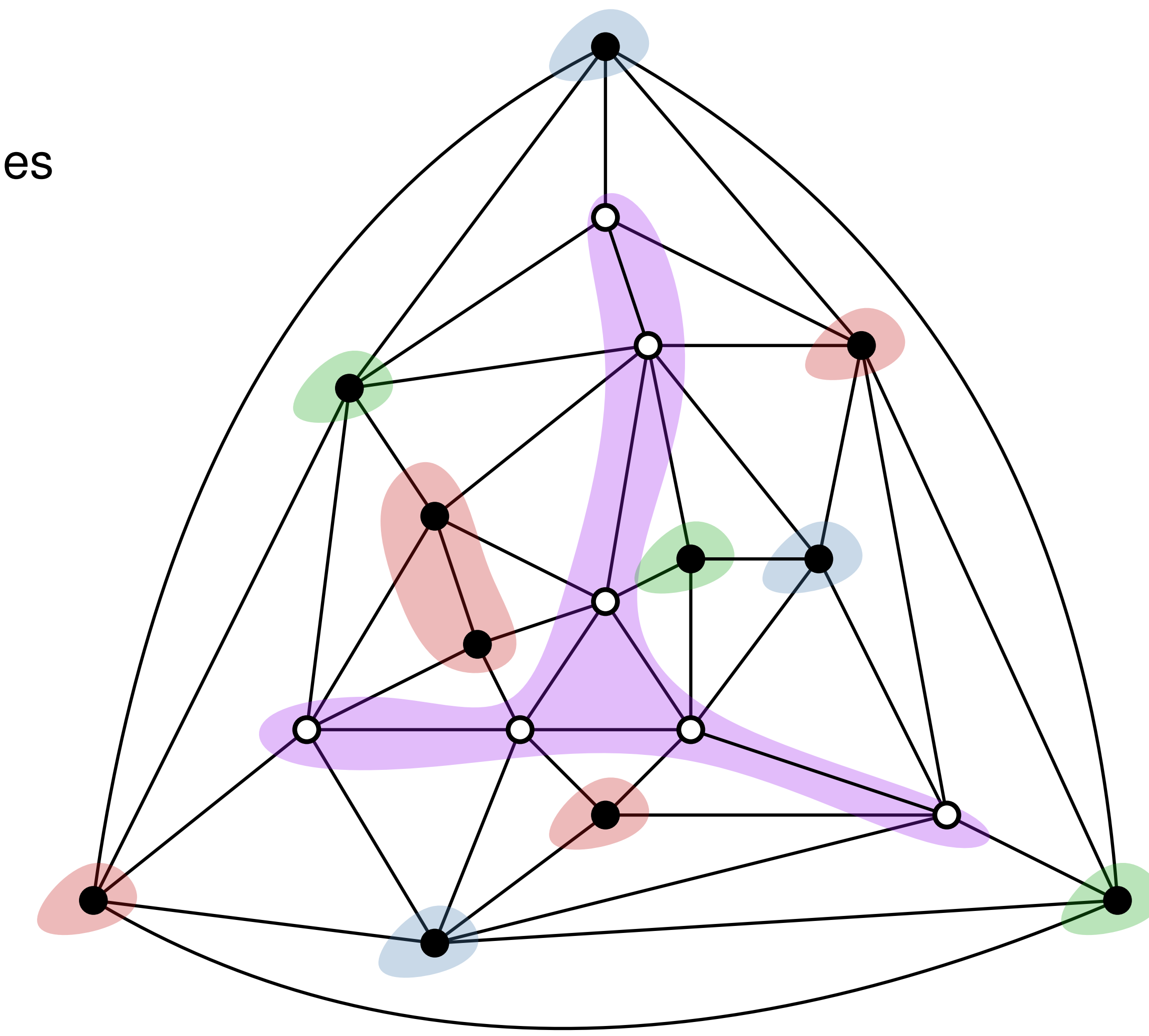

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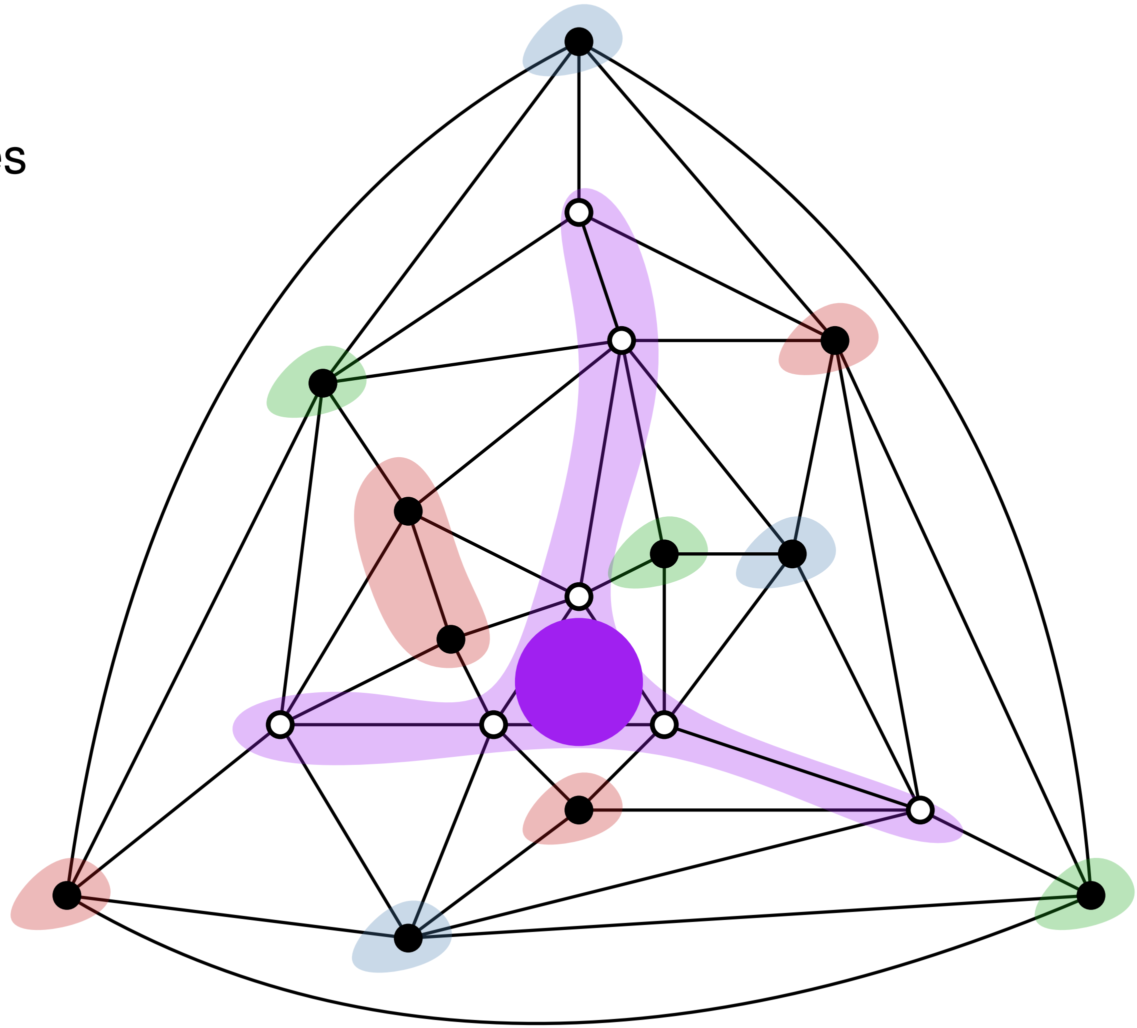

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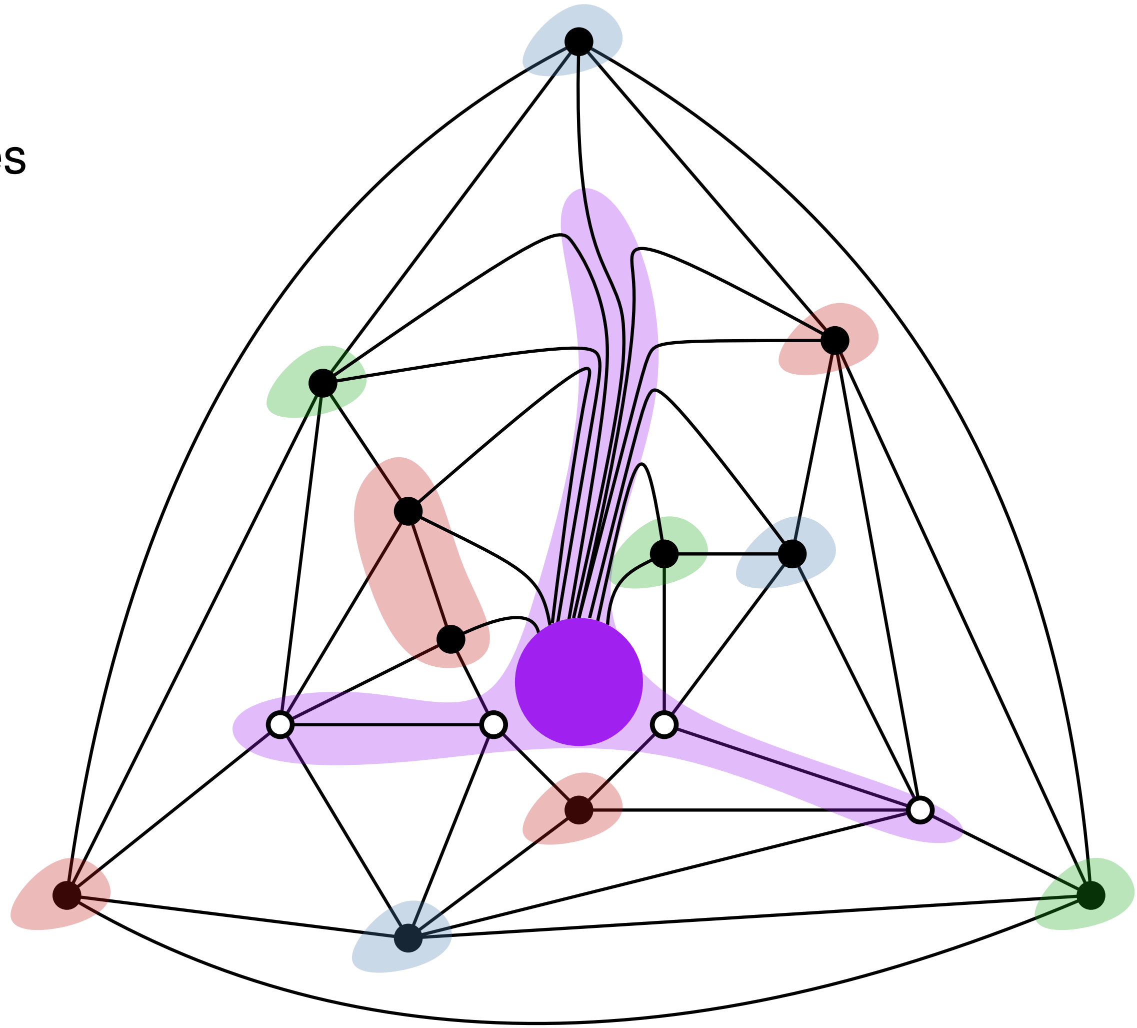

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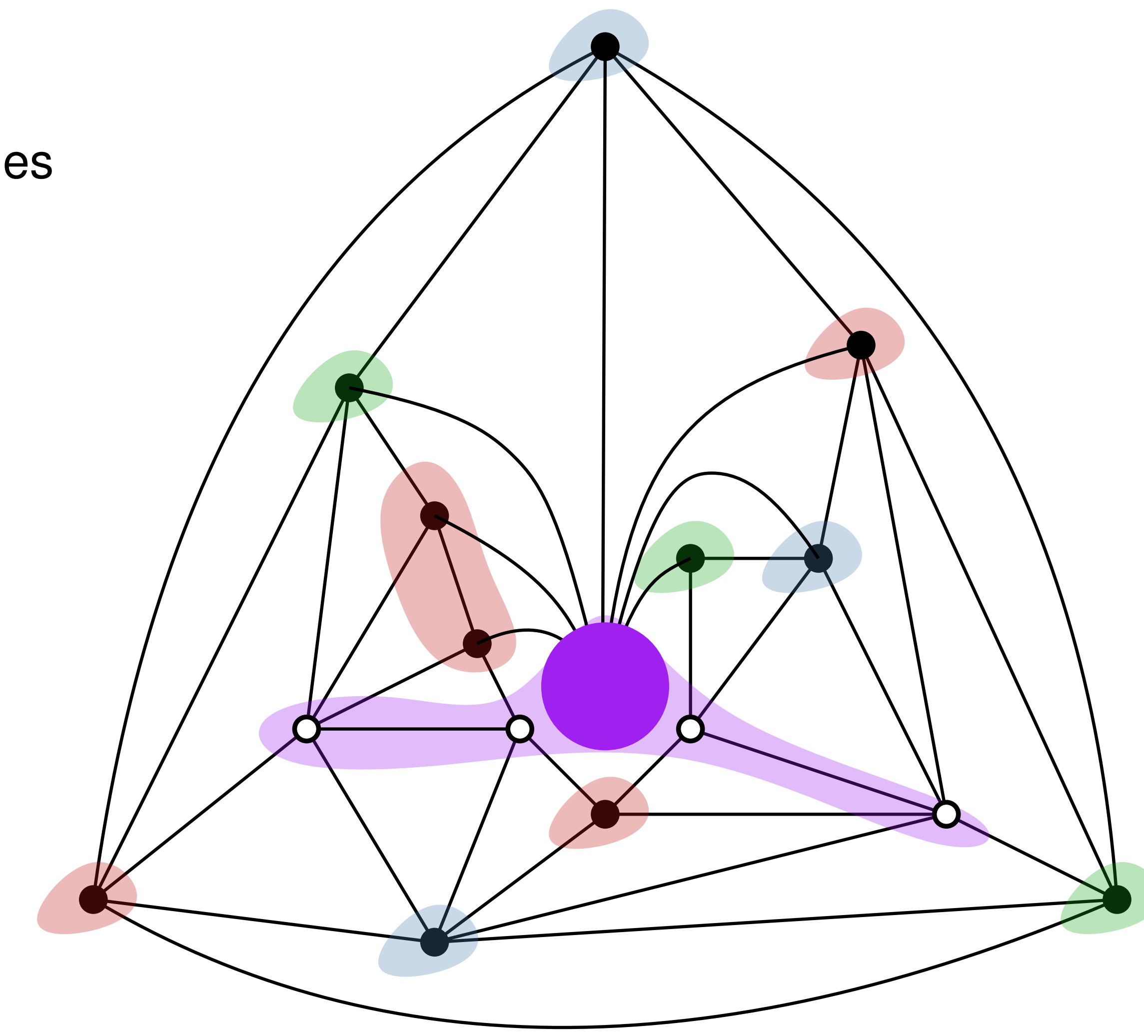

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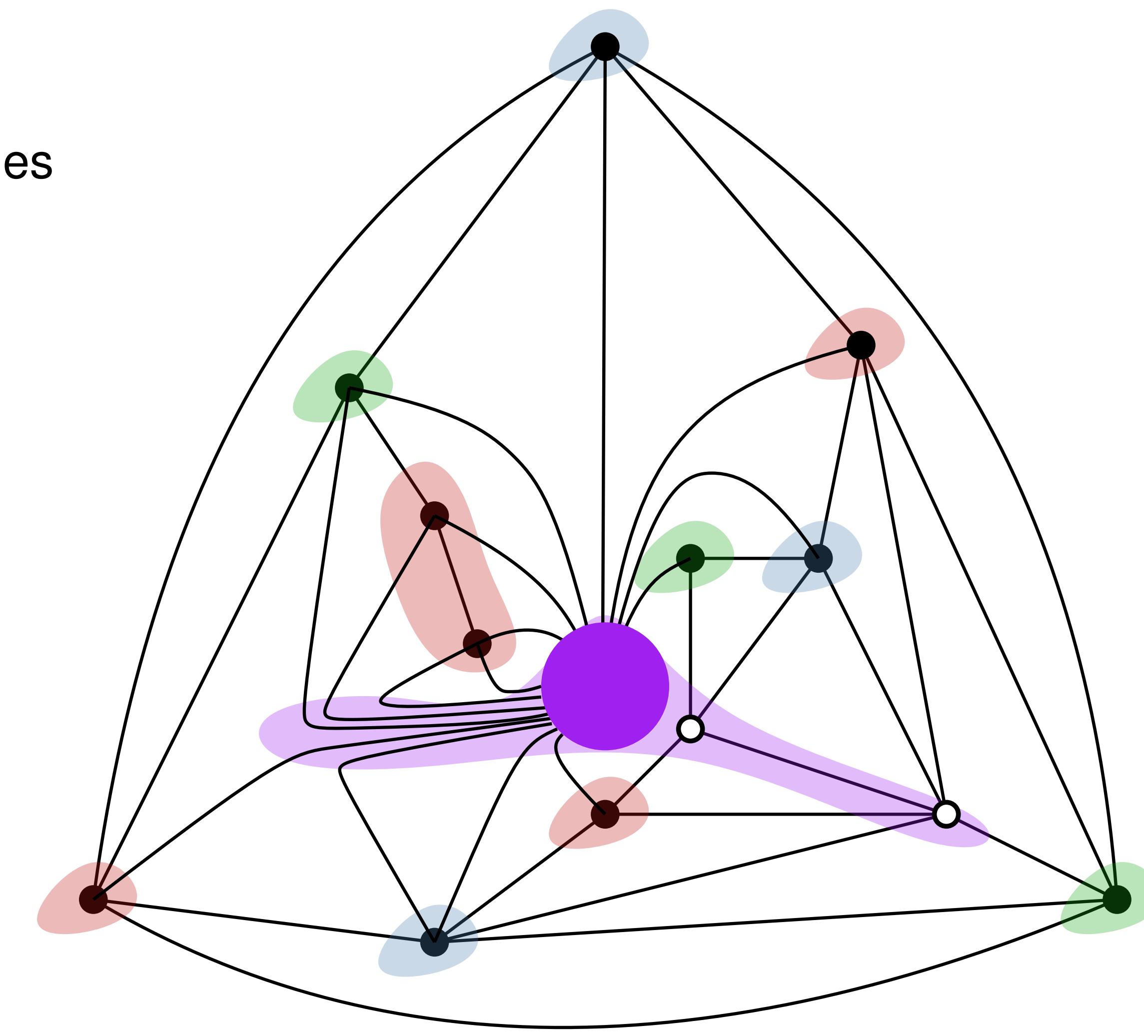

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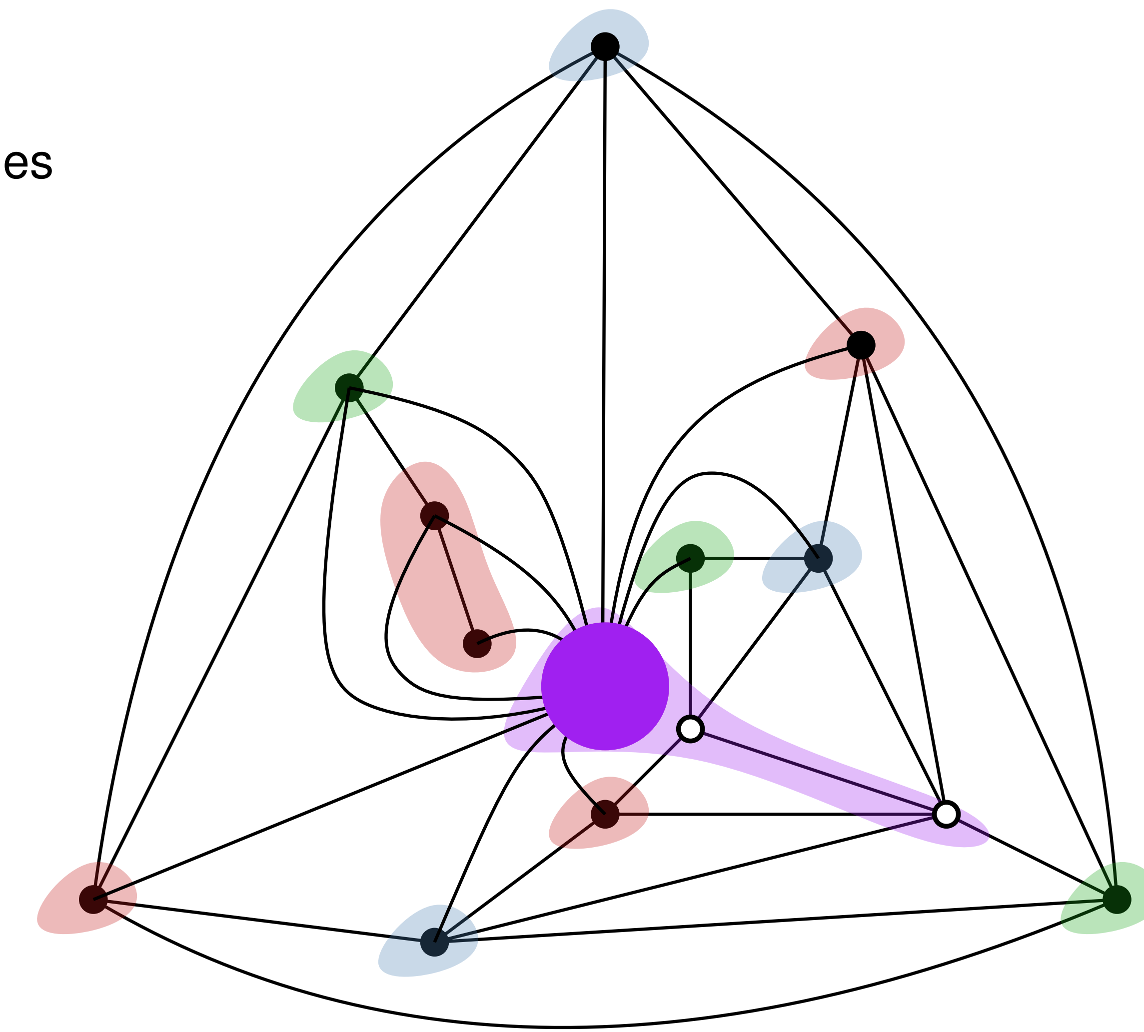

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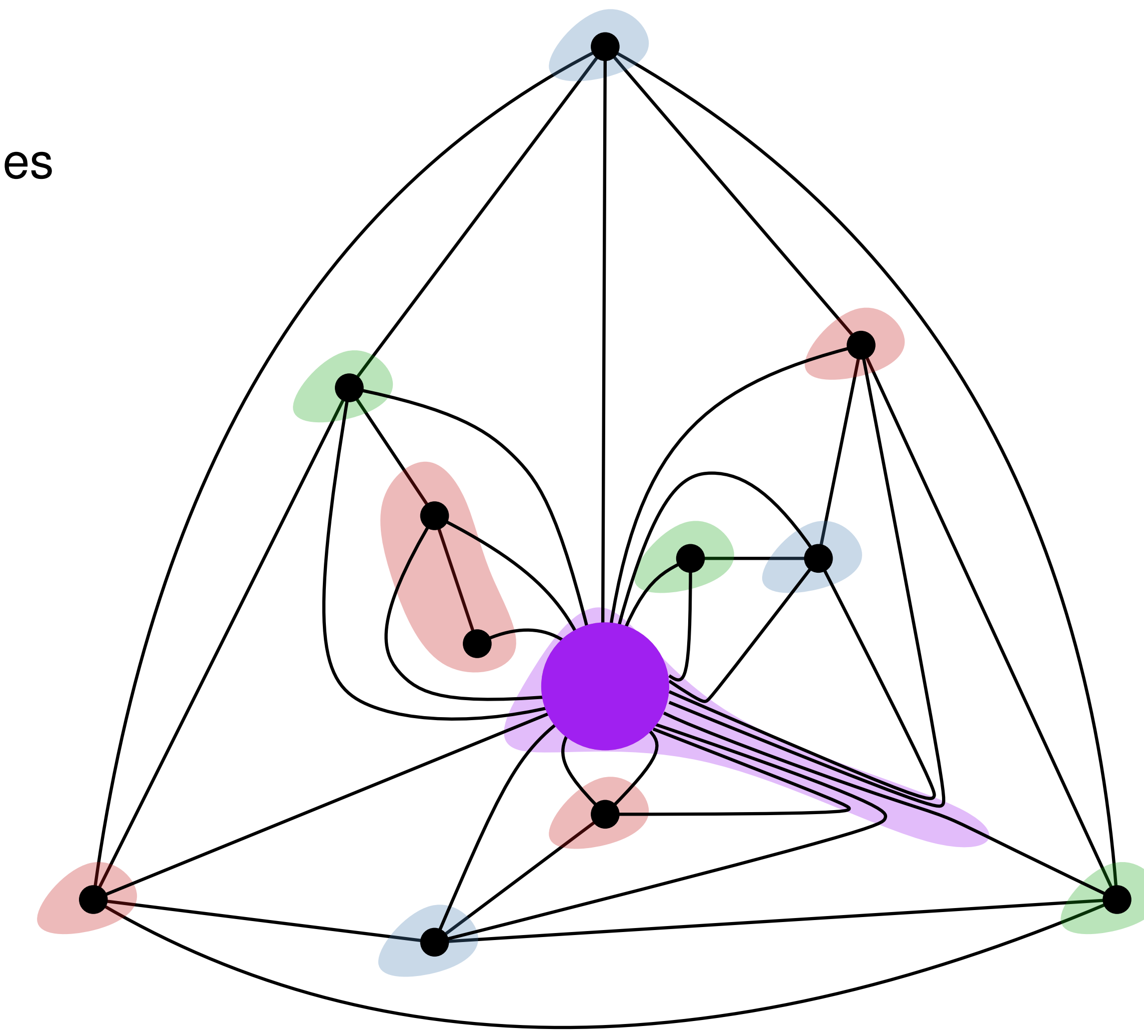

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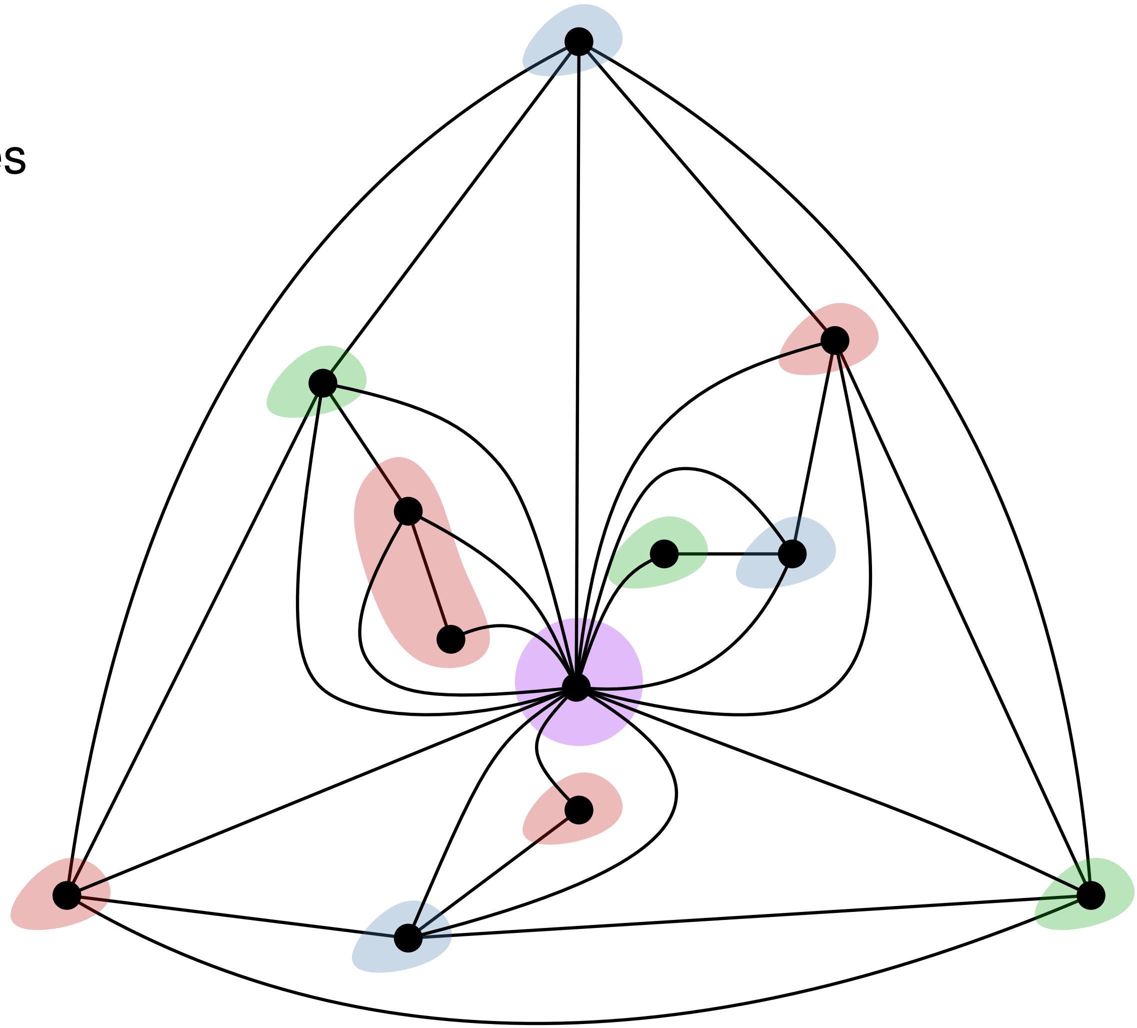

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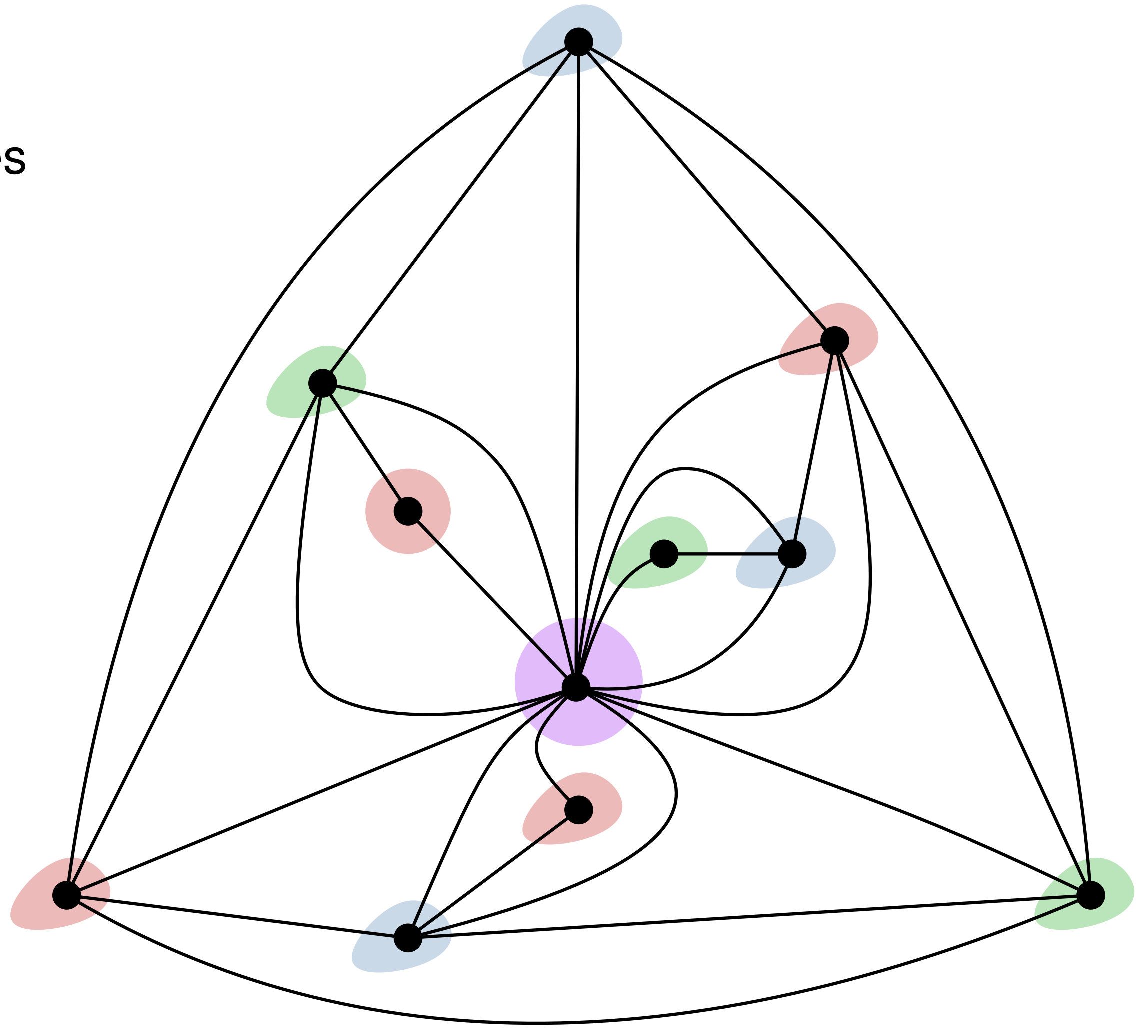
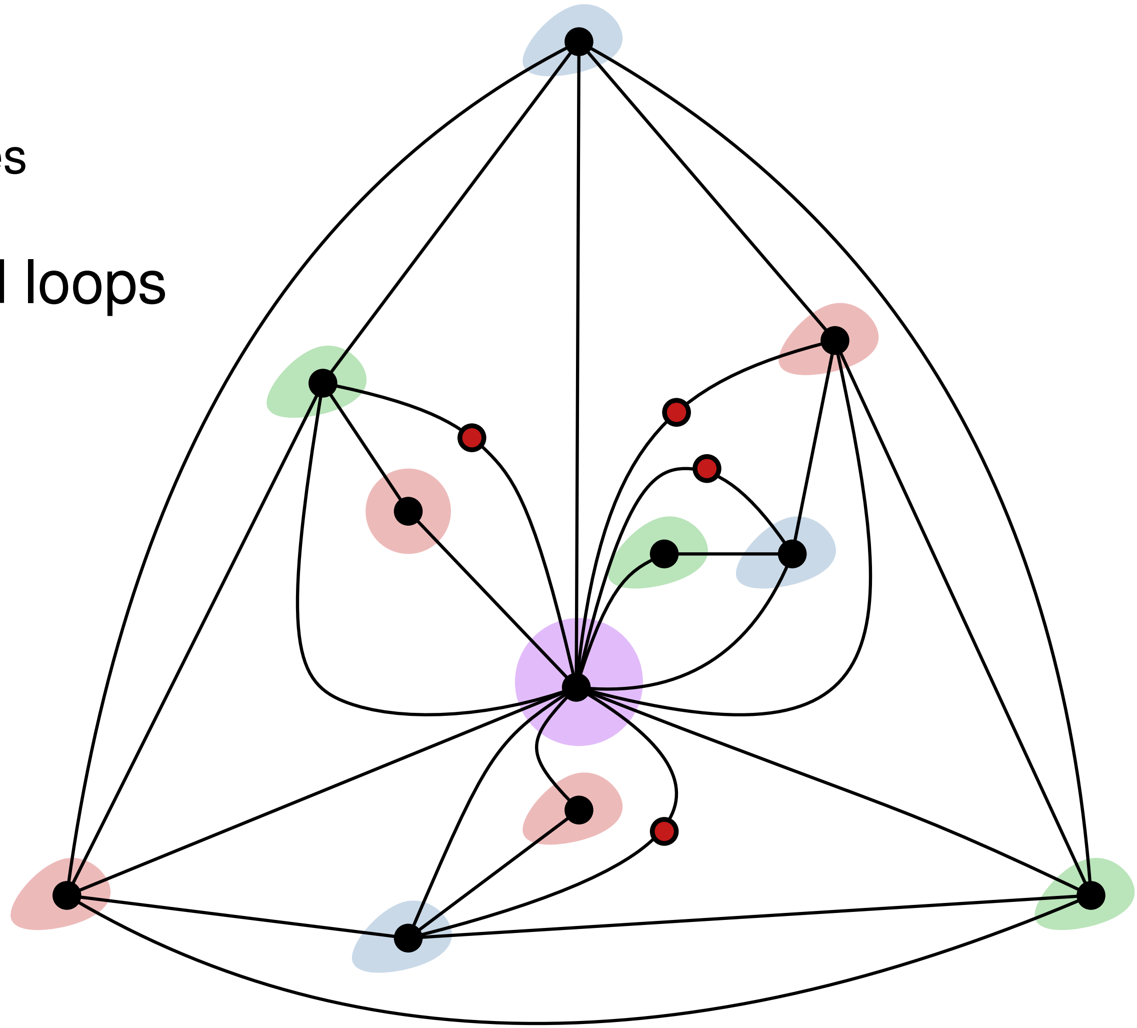


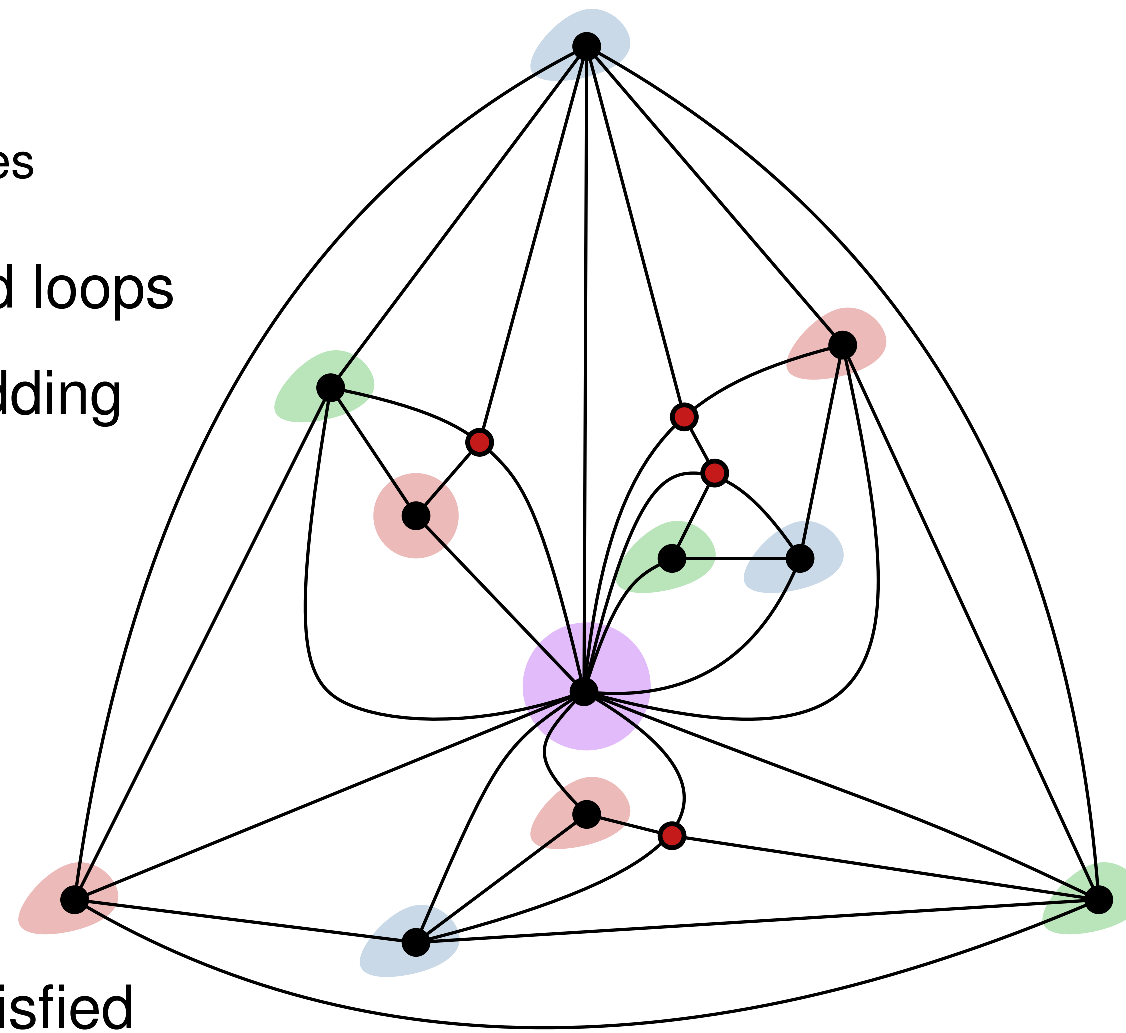
Improvement

- Contract tripods in G
  - preserve embedding
  - remove homotopic parallel edges
- Subdivide parallel edges and loops



Improvement

- Contract tripods in *G*
  - preserve embedding
  - remove homotopic parallel edges
- Subdivide parallel edges and loops
- Extend by preserving embedding



- All three restrictions are satisfied
- The paths can be ordered with no 3-rainbows

Summary

Improved bound:

$$qn(G) \leq 3 \cdot (3 \cdot 3 + 2 \cdot 2) + 3 = 42$$

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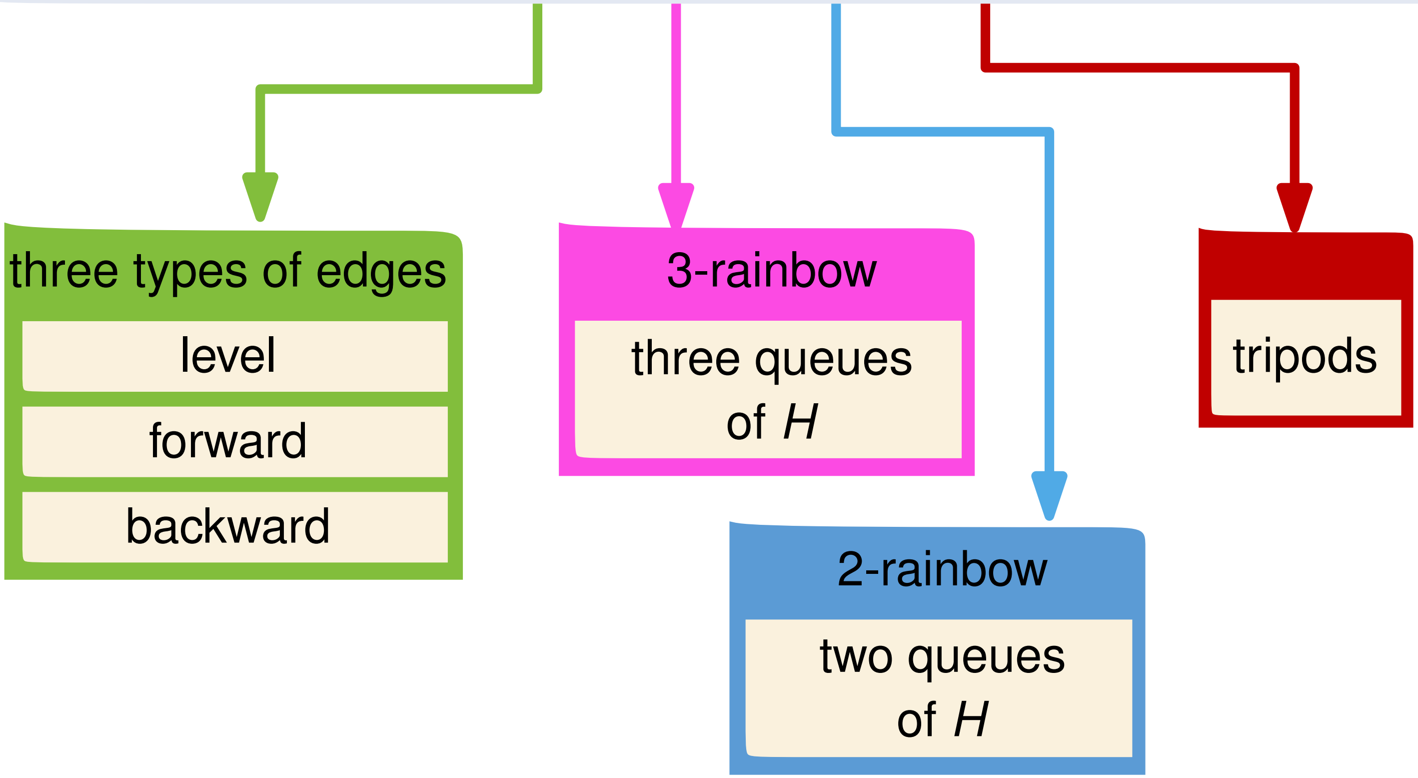

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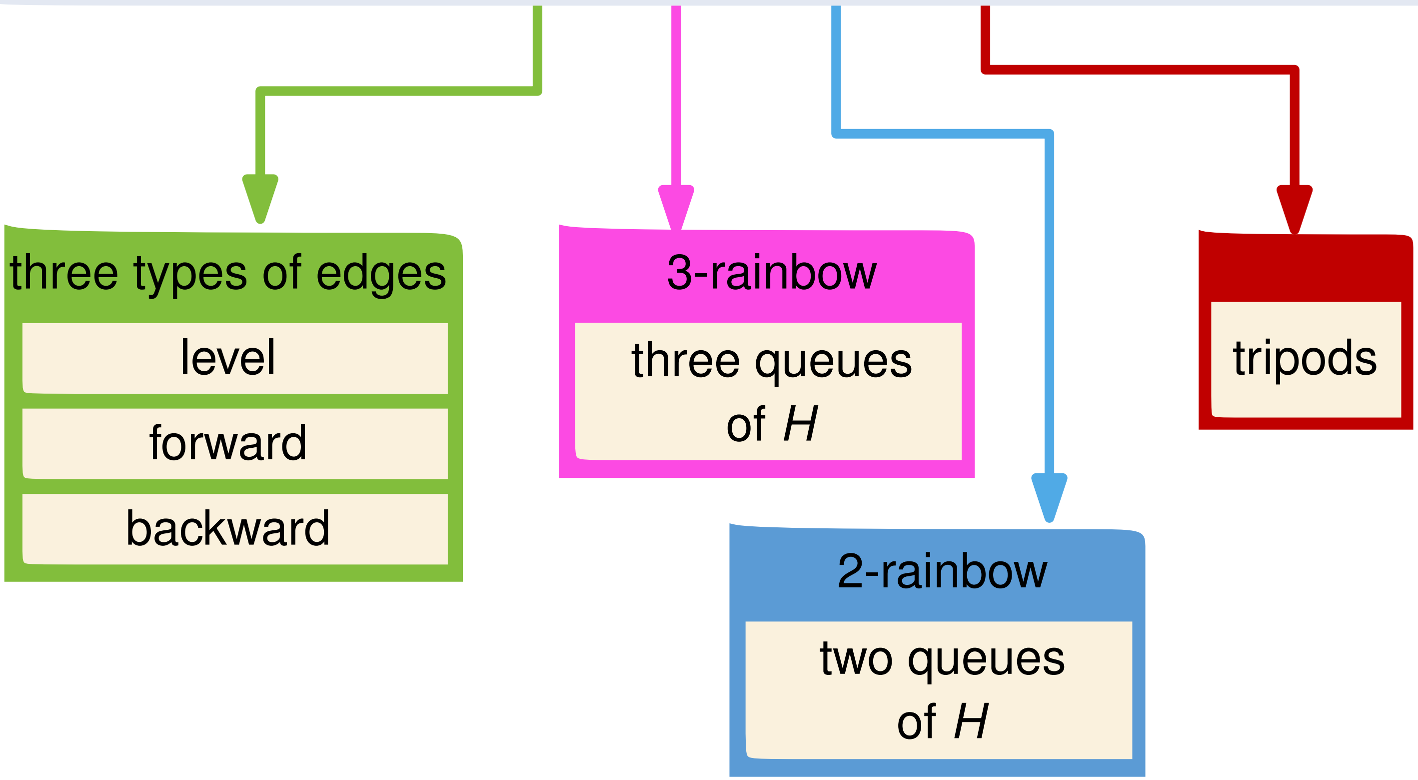
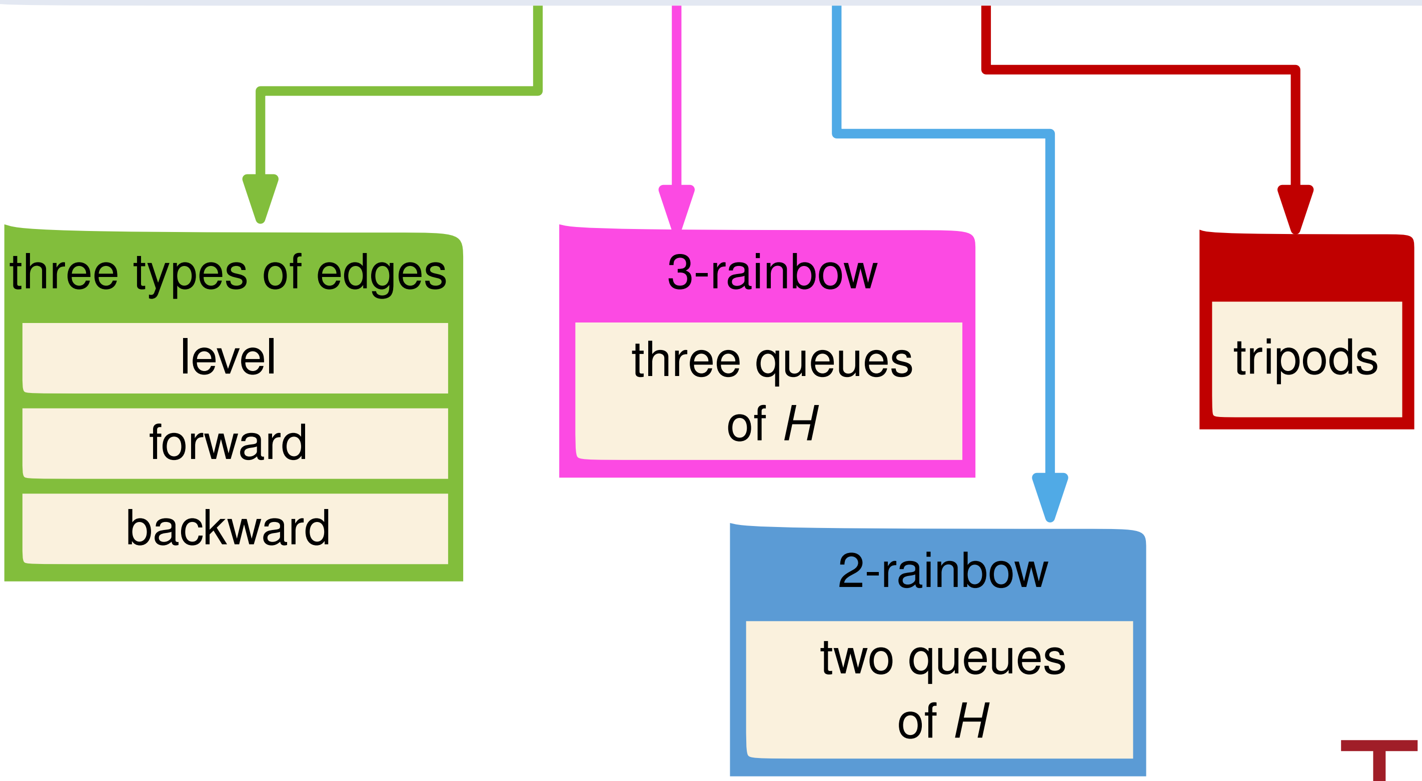


- Further reduce
- Subfamilies

Summary

Improved bound:

$$qn(G) \leq 3 \cdot (3 \cdot 3 + 2 \cdot 2) + 3 = 42$$



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Thank you